Math 137: Algebraic Geometry

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0 Preface

This class is from Spring 2023. Meeting times are Monday and Wednesday from 10:30-11:45am. There is no designated textbook, but we'll be consulting some standard texts for inspiration, like Hartshorne, Shafarevich, etc.

Problem sets will be assigned weekly and due every Wednesday. Popa's office hours will be Wednesdays from 12-1pm. Office hours and section times for the course assistants can be found on Canvas. There is no midterm or final exam for this class!

All errors in these notes are attributed to me. I am already aware of a few unfinished arguments which need some patching. (I couldn't type fast enough.) If you see anything wrong or unclear, please let me know at hahnlheem@gmail.com!

1 01/23 - The Story of Algebraic Geometry

1.1 Historical Motivation

Lots of text here, but do not worry – the "math" will come in due time :)

It's hard to give a good definition of algebraic geometry, but the name suggests that we're constructing some bridge between algebra and geometry. This is a lot more familiar of a notion than you may think: the equation $x^2 + y^2 = 1$ is purely an algebraic equation, but we get a really nice perspective when we think of the equation as carving out a circle in $\mathbb{R}^2$. The circle, in other words, is the set of solutions of the equation $x^2 + y^2 = 1$ in the reals.

So let's find solutions to polynomial equations more generally. We understand very well what happens in the settings for polynomials $f(x) = 0$ in just one variable. If $\deg f = 1$, it's just a linear equation, which is elementary school stuff. $\deg f = 2$ is just solving a quadratic, also easy. A little bit more difficult is when $\deg f$ is 3 or 4, but there do exist formulae for the solutions which use only the fundamental operations and the coefficients of f. Interesting stuff happens beyond 4 though: Galois theory and the Abel-Ruffini theorem tells us that there is no such explicit formula for a general $\deg f \geq 5$ polynomial.

Let's shift our attention to polynomials $f(x, y) = 0$ in two variables, or more generally a system of equations $f_1(x, y) = \cdots = f_m(x, y) = 0$ of polynomials in two variables. Our goal is to take this very algebraic setting and relate it to the geometry that arises from considering the solutions of the given system.

The reals are annoying in terms of finding solutions; we know that in the case for one variable, you have quadratics which don't have any real roots at all. Meanwhile, any polynomial in $\mathbb{C}$ is guaranteed to factor into linear terms by the Fundamental Theorem of Algebra. ($\mathbb{C}$ is algebraically closed field, $\mathbb{R}$ is not.) So one major advancement in this, if we may call it, "algebraic geometry" problem, is to work over $\mathbb{C}$ rather than over $\mathbb{R}$.

As a simple example, if we consider the solutions of $f(x, y) = x^2 + y^2$ over $\mathbb{R}$, we get a single point $(0, 0)$, whereas over $\mathbb{C}$, we get two lines $x - iy = 0$ and $x + iy = 0$, which is a completely different and more interesting story.

The other major advancement to this setting in two variables is to consider the solutions not in $\mathbb{C}^2$ or $\mathbb{R}^2$, but over $\mathbb{P}^2$, the projective plane. We'll get to this in due time.

1.2 Formalizing Ideas

Let k be some field (usually we'll work with something like $\mathbb{C}$, sometimes $\mathbb{R}$). Let $f \in k[X,Y]$. Then, we define the **zero locus of f** as exactly that: the set of zeroes of the polynomial. We notate

$$C = Z(f) = \{(x,y) \mid f(x,y) = 0\} \subseteq k^2.$$

Z stands for the zero locus. Sometimes we notate this as C, as this is a **plane algebraic curve**. We also often call $k^2 = \mathbb{A}^2$ as the **affine plane**.

Example 1.1

Let $f(X,Y) = X^2 + Y^2 + 1$. Over $\mathbb{R}$, it has no solutions, but over $\mathbb{C}$, this is a conic equation, mapping out a circle in the affine plane.

Again, the above example highlights the belief that it is better to work in $k = \overline{k}$ an algebraically closed field. In fact, if $f \in k[x,y]$ where $k = \overline{k}$, then $Z(f)$ is **always an infinite set**. (Convince yourself that this is true!) So the geometry that arises is a lot more interesting than just having a finite collection of points.

1.3 Motivating Projective Space

We mentioned before that the second major advancement in considering our algebraic geoemtry problem in two variables is considering equations over the projective plane rather than $\mathbb{R}$, or even $\mathbb{C}$.

Here's an issue that arises in $\mathbb{R}$ or $\mathbb{C}$ but is resolved in the projective plane. Consider a line and a circle over $\mathbb{R}$. The line may be disjoint from the circle, tangent to the circle, or secant to the circle, corresponding to $0, 1, 2$ intersection points, respectively. We don't like this inconsistency. Over $\mathbb{C}$, we run into a similar problem when considering intersections of two circles: they can be disjoint, tangent, intersecting at two points, or intersecting at four points.

But if we introduce a "point at infinity" for each direction in $\mathbb{C}^2$ through 0, we get a projective plane $\mathbb{P}^2$, which will resolve our problems. We'll define the projective plane more rigorously later on, so if this doesn't sound too convincing at the moment, don't worry.

But just to give a preview of how nice the theory of intersections becomes in the projective plane, we have a really nice characterization given by Bezout's Theorem.

Theorem 1.2 (Bezout's Theorem)
Let $\deg f = d$ and $\deg g = e$. If $C = Z(f)$, $D = Z(g)$, and one is not contained in the other, then the number of intersection points of C and D in $\mathbb{P}^2$ is always equal to $d \cdot e$, counting multiplicities.

As an example, two conic ($\deg f = 2$) equations will always intersect in $2 \cdot 2 = 4$ points, thus resolving our previous problems in $\mathbb{R}$ and $\mathbb{C}$.

1.4 Aside: Non-algebraic Fields

Even though we can easily describe $\mathbb{R}$ and $\mathbb{C}$ geometrically, the study of algebraic geometry goes beyond these fields, especially in the number theory setting. For example, $\mathbb{Q}$ is relatively "ugly" geometrically, but considering the integer solutions to $X^n + Y^n = Z^n$ is equivalent to finding the solutions to $f(X, Y) = X^n + Y^n - 1$ over $\mathbb{Q}$. As you may know, Fermat's Last Theorem characterizes the solutions to this two-variable polynomial, the proof of which is largely written in the language of algebraic geometry. It is also immensely common in number theory to find solutions to equations over finite fields $\mathbb{F}_p$ – ask me if you'd want to see this kind of stuff!

For a brief glimpse of the historical story of algebraic geometry, mathematicians like Zariski and Weil made algebraic geometry a lot more "algebraic" in the early 20th century, so the study of AG became intricately connected with commutative algebra. (Thus 221 is a really good class to take if you want to study more algebraic geometry!) Starting around 1950, some really big brained giants like Serre, Grothendieck, Mumford somehow came up with a radically different way to formulate the notions tossed around in algebraic geometry, giving birth to the modern perspective, which is what most mathematicians use today. Take 232a/b for a fun time!

On that note, although 137 only covers the classical perspective and is thus not really used much these days, it is extremely important to help understand the modern perspective, which oftentimes loses the geometric motivation in all its technicalities and abstractness.

1.5 Plane Algebraic Curves

We now dive into a few definitions.

Definition 1.3 (Affine plane)**.** Let $k = \overline{k}$ be an algebraically closed field. Then, the **affine plane** is

$$\mathbb{A}^2_k = \{(x, y) \mid x, y \in k\}.$$

Oftentimes, when k is well-understood, we notate as $\mathbb{A}^2$.

We saw earlier that a nonconstant polynomial f gives rise to a plane algebraic curve $C = Z(f) \subseteq \mathbb{A}^2$.

Definition 1.4 (Degree of curve)**.** The degree of the plane algebraic curve $C = Z(f)$ is given by $\deg(f)$.

You'll see over time that this class is a lot more algebraic than geometric (thanks to the work done in the early 20th century), so we'll be studying a lot of commutative algebra in this class. This starts now:

Fact 1.5 (Polynomial Rings are UFDs)**.** The ring $k[X_1, \ldots, X_n]$ is a UFD.

Proof. (Sketch) Appears in any ring theory class, but this follows from k being a field, hence a UFD, and the fact that R is a UFD implies $R[X]$ is a UFD. (Use Gauss's Lemma.) □

The above fact means that we can always factor any $f \in k[X_1, \ldots, X_n]$ as $f = \alpha \cdot f_1^{n_1} \cdots f_p^{n_p}$, where each f_i is irreducible and $\alpha \in k$.

Definition 1.6 (Irreducible components)**.** Let $f \in k[X_1, \ldots, X_n]$ factored as above. Then, we say $C_i = Z(f_i)$ are the **irreducible components** of C. Each C_i is an **irreducible curve**.

It is useful to describe (irreducible) curves using parameterizations. To illustrate with an example, the curve given by $f(X, Y) = Y - X^2$ can be parameterized by $t \mapsto (t, t^2)$. In particular, this gives us a very nice isomorphism

$$\begin{aligned} \mathbb{A}^1 &\xrightarrow{\sim} C \\ t &\mapsto (t, t^2). \end{aligned}$$

Plane algebraic curves like above that exhibit parameterizations in one variable are special enough to warrant their own name:

Definition 1.7 (Rational)**.** An affine plane curve $C = Z(f)$ is **rational** if there exists rational functions $u(t), v(t)$ in one variable such that $f(u(t), v(t)) = 0$ for almost all t.

Remark 1.8. The isomorphism in the case of $f(X, Y) = Y - X^2$ is strong and often not the case, even when the curve is rational. Usually, the best we can do is construct an isomorphism between the affine line and the curve "almost everywhere." We'll see this more explicitly later, but we give an example below.

Example 1.9 (Rational curves)

By definition, every line is rational. It turns out that every conic is also rational. To give a picture of this, take a conic C and a line $\ell \cong \mathbb{A}^1$ disjoint from C. Choose any point P on C. Then, the line connecting P with any point $Q \in \ell$ intersects C at a unique point, and furthermore, every point on C (except P) is the intersection of some line between P and a point on ℓ. Thus, then we get an almost-isomorphism between C and $\mathbb{A}^1$, with the point P hindering this from being an isomorphism.

Let's write this out explicitly. Let $C = Z(f)$ be a conic and fix some $(x_0, y_0) \in C$. Consider now the family of lines $\ell_t : y - y_0 = t(x - x_0)$ parametrized by t. Then, $C \cap \ell_t$ is given by $f(x, t(x - x_0) + y_0) =: f(x, g(x)) = 0$, which is a quadratic in x for almost all t. One root is x_0; call the other x_t. Working out the details, one can find

$$\begin{cases} x_t = -x_0 - A(t) \\ y_t = y_0 + t(x_t - x_0) \end{cases}$$

as a rational parameterization, where $A(t)$ here is the coefficient of x in $f(x, g(x))$.

2 01/25 - Map Between Curves

We begin with the setup from last time. Take some $f(x, y) \in k[x, y]$ where $k = \overline{k}$ is algebraically closed. Denote the plane algebraic curve $C = Z(f) \subseteq \mathbb{A}^2$. Recall from last time that C rational means there exists rational functions $u(t), v(t)$ such that $f(u(t), v(t)) = 0$. Lines and conics are both examples of rational functions.

Now suppose C is irreducible (i.e. f is irreducible), and suppose $P(X, Y), Q(X, Y) \in k[X, Y]$ such that $f \nmid Q$. Consider the rational function $w(X, Y) = \frac{P(X,Y)}{Q(X,Y)}$. This is not always well-defined on all of C as $Q(X, Y)$ could vanish somewhere on C. However, this poses minimal problems.

Exercise 2.1. Suppose $f, g \in k[X, Y]$, f is irreducible, and $f \nmid g$. Then, $Z(f)$ intersects $Z(g)$ in at most finitely many points.

By the exercise, the zero locus of Q intersects C in only finitely many points, which means $w(X, Y)$ is defined on all but finitely many points of C.

Definition 2.2 (Function field of C)**.** Let $C = Z(f)$. The **field of rational functions** on C (or the **function field** of C) is

$$k(C) = \left\{ w = \frac{P}{Q} : f \nmid Q \right\} / \sim,$$

where $\sim$ is given by $P_1/Q_1 \sim P_2/Q_2 \iff f \mid P_1Q_2 - P_2Q_1$.

The $f \mid P_1Q_2 - P_2Q_1$ condition makes sense because if two rational functions are to be considered the same on C, then their difference must vanish on C.

We know rational curves can be parameterized by a single variable t. Let's give a hint at where this leads: because we're still working in two variables, we know that $k(C)$ is generated by the rational functions X, Y. But C gives us an algebraic relation $f(x, y) = 0$ between the two variables, implying $\operatorname{trdeg}_k k(C) = 1$. This leads us to a nice characterization of their function fields:

Proposition 2.3
C is rational iff $k(C) \cong k(t)$.

Proof. Suppose $k(C) \cong k(t)$. Consider the isomorphism $\Phi : k(C) \to k(t)$, and call the images $\Phi(x) = u(t)$, $\Phi(y) = v(t)$. Then, since $f(x, y) = 0$, taking the function under Φ gives $f(u(t), v(t)) = 0$, so C is rational. For the other direction, we will first define a homomorphism

$$\begin{aligned}\Phi &: k(C) \to k(t) \\ w(X, Y) &\mapsto w(u(t), v(t)) = \frac{P(u(t), v(t))}{Q(u(t), v(t))},\end{aligned}$$

where $u(t), v(t)$ satisfy $f(u(t), v(t)) = 0$. (Such a parameterization exists because C is rational.) By Exercise 2.1, the denominator $Q(u(t), v(t))$ of $w(u(t), v(t))$ is nonzero for almost all t. Further, Φ is well-defined, as we map $X \mapsto u(t)$ and $Y \mapsto v(t)$ and extend linearly. We now invoke a theorem from field theory, which says there are no intermediate fields in a purely transcendental field extension of degree 1:

Theorem 2.4 (Lüroth)
Suppose $k \hookrightarrow L \hookrightarrow k(t)$ with $L \neq k$. Then, $L \cong k(t)$.

Since $k(C)$ trivially contains k and any nontrivial field map is injective, we can invoke Lüroth's Theorem on $k \hookrightarrow k(C) \overset{\Phi}{\hookrightarrow} k(t)$ to get the desired result. □

Note that this highlights a pretty intricate connection between field theory and geometry. More generally, if L is some purely transcendental field extension over k with $\operatorname{trdeg}_k L = 1$, say $L = K(X)$, then a projection map $k(X, Y) \twoheadrightarrow L$ forces some algebraic relation $f(X, Y) = 0$. We can then view $L \cong k(Z(f))$, giving L geometric significance. There is some correspondence between field extensions L/k with $\operatorname{trdeg}_k L = 1$ and affine curves.

Exercise 2.5. Suppose C is rational, parametrized by $u(t), v(t)$ (given from $\Phi : k(C) \to k(t)$). Then, except for a finite set of points on C, every $(x_0, y_0) \in C$ has a *unique* representative $(x_0, y_0) = (u(t_0), v(t_0))$.

2.1 Singular Points

Now we're looking at local behavior of these curves. When studying curves and stuff for the first time in high school calculus, one often started out by reciting names to a bunch of edge case scenarios. For instance, a curve may have a node, or a cusp. We'll define these more rigorously:

Definition 2.6 (Singular points). Let $f \in k[X,Y]$ be nonconstant and $C = Z(f) \subseteq \mathbb{A}^2$. A point $P \in C$ is **singular** if

$$f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0.$$

Otherwise, we say P is **nonsingular**, or **smooth**. If C is nonsingular for all points, we call it a smooth curve.

Let's see how this definition makes sense. For convenience, we'll suppose $P = (0,0)$ (we can always make it as such by translation). The partial derivatives being 0 means that $f(X,Y) = aX^2 + bXY + cY^2 +$ higher order terms.

Example 2.7 (Singular conics)

In the case where f is conic, i.e. $\deg f = 2$, then there are no higher order terms, so C singular at $P = (0,0)$ iff $f(X,Y) = aX^2 + bXY + cY^2$, i.e. f is a product of linear factors (we're assuming here $k = \overline{k}$). There are two cases: $f = gh$ where $g \neq h$, or $f = g^2$. Geometrically, the former is the union of two lines, while the second is a "double line." Giving a better description of the double line leans more into scheme theory territory, so we stop here.

Example 2.8 (Some cubics: (almost) elliptic curves!)

(These are all examples of curves that would be elliptic curves if there weren't any singular points.) Here, we look at two different kinds of singular points. Consider the curve $f = y^2 - x^3$. I'm bad at putting diagrams here, but if you put this into W-A or something, you'll see a picture of a curve entirely in the region $x \geq 0$, and two curves intersect "sharply" at the origin.

In contrast, the curve $f = y^2 - x^2 - x^3$ is still singular at $P = (0,0)$ (take the partial derivatives), but the drawing looks, crudely described, like a fish with tail extending to infinity. The point is that the curve intersects itself at the origin. Locally, this picture looks like two lines intersecting, like in the situation $f = gh$ for linear $g \neq h$, but the global picture is different.

Similar to how roots can have multiplicity, there is also a notion of multiplicity for singular points, which corresponds to our usual notion of multiplicity of root and gives us an indicator of "how singular a point is."

Definition 2.9 (Point of multiplicity). $P = (0,0)$ is a **point of multiplicity** r on C if the lowest degree of a nontrivial term in f is r, i.e. $r = \text{mult}_C(P)$.

Two sanity checks: $P \in C$ is singular iff $\text{mult}_C(P) \geq 2$, and $\text{mult}_C(P) \leq \deg(C)$ for all $P \in C$. Convince yourself that both are true!

Although singular points do sometimes bring about interesting math on their own, we only want them in very little moderation, since they are cumbersome to deal with. Thankfully, this is always the case!

Proposition 2.10

An irreducible plane curve has only finitely many singular points.

Proof. Let $C = Z(f)$, and denote $C_x := Z\left(\frac{\partial f}{\partial x}\right)$. If P is singular, then $P \in C \cap C_x$. If $|C \cap C_x| < \infty$, we are done, else suppose that $|C \cap C_x| = \infty$. Invoking our exercise again, this implies $f \mid \frac{\partial f}{\partial x}$, which forces $\frac{\partial f}{\partial x} = 0$, since $\deg f > \deg \frac{\partial f}{\partial x}$. By symmetry, $\frac{\partial f}{\partial y} = 0$.

If $\operatorname{char} k = 0$, then we may conclude that f is constant, a contradiction. If $\operatorname{char} k = p$, then we may write $f(X, Y) = g(X^p, Y^p)$. By the freshman's dream $(a+b)^p = a^p + b^p$, we can write $g(X^p, Y^p) = (h(X, Y))^p$, which means f is reducible, a contradiction as well. $\square$

2.2 Rational Maps

We're going to tie up a lot of ideas that's been floating around today in this final definition, which will become a very familiar notion by the end of this class.

A **rational map** is simply a map which is a rational function on each component. Let $C = Z(f) \subseteq \mathbb{A}^2$. Let $u, v \in k(C)$ be two rational functions. Then, we can construct a "map"

$$\begin{aligned} C &\dashrightarrow \mathbb{A}^2 \\ P &\mapsto (u(P), v(P)). \end{aligned}$$

The $\dashrightarrow$ instead of $\to$ is intentional: the name is a bit misleading because both $u(P), v(P)$ are not necessarily defined on all of C; at best, this is defined for all but finitely many points. Realizing the image of the "map" as a curve itself, we have:

Definition 2.11 (Rational Maps)**.** Let $C = Z(f), B = Z(g) \subset \mathbb{A}^2$ with f parameterized by $u(t), v(t)$ and $g(u(t), v(t)) = 0$. Then, we get a "map" $C \dashrightarrow B \subseteq \mathbb{A}^2$. We call such a "map" a **rational map**.

Example 2.12

We already saw an example of this! Take a conic (say a circle/ellipse) and a line ℓ disjoint from the conic. Choose any point P on the conic. Then, we have a rational map where, given some $Q \neq P$ on the conic, Q maps to the intersection of ℓ and the line connecting P and Q. Note this is defined for all of C except P itself. If you give the line and conic explicit formulae, it is clear that this map is given by rational functions.

When the map has an inverse, it is called birational.

Definition 2.13 (Birational)**.** A rational map φ is called **birational** if $\exists \psi : B \dashrightarrow C$ rational map such that $\varphi \circ \psi = \text{id}_B$ and $\psi \circ \varphi = \text{id}_C$ whenever defined. In this case, B and C are also called **birational**.

Thus, we may summarize a previous result very succinctly now: C **is rational iff** C **is birational to** $\mathbb{A}^1$.

3 01/30 - Plane Curves, continued

This will be the last lecture of discussing these algebraic geometry phenomena via the specific case of plane curves, i.e. after today, we'll venture into situations with more than just two variables.

Consider $C = Z(f)$ and $D = Z(g)$ in $\mathbb{A}^2$, and suppose C is irreducible, $C \not\subset D$.

Definition 3.1 (Intersection multiplicity)**.** Let $p \in C$ be a smooth point. The **intersection multiplicity** of C and D at p is defined as the multiplicity of p as a zero of $g|_C$. We denote it as $i_P(C, D)$.

This may not be entirely clear of a definition at first. We do know the case when $f(x, y) = y + h(x)$, i.e. when we can express y in terms of x, in which case $g(x, y)|_C = g(x, -h(x))$ and we can count multiplicity of the root. We'll define the intersection multiplicity in the general case later, but to complete the discussion in two variables, one may look at Shafarevich p. 13-14.

For now, though, we'll only consider the case when C is a line, in which case our definition is clear. Let's consider $C = (y = 0)$ and $D = (y - x^2 = 0)$. Graphically, C is a tangent line to the parabola D; we expect the intersection multiplicity to be 2. Indeed, $i_P(C, D)$ is the multiplicity of 0 as a root of x^2, which is 2.

As one knows from calculus, we can always find a tangent line at a smooth point. We'll compute this explicitly.

Claim 3.2 (Tangent line to C at p)**.** Let $p = (x_0, y_0) \in C = Z(f)$ is a smooth point. Then, there exists a unique line L through p at $i_P(L, C) \geq 2$.

We then define the **tangent line at** p as the above unique line.

Proof. A line L passing through $p = (x_0, y_0)$ can be parameterized by $x = x_0 + \lambda t$ and $y = y_0 + \mu t$, where $t \in k$. ($\lambda, \mu \in k$ are fixed.) We may write

$$f(x, y) = a(x - x_0) + b(y - y_0) + g(x, y)$$

where $g(x_0, y_0) = 0$ and g consists of terms of deg ≥ 2. We know at least one of a, b is nonzero since p is smooth. Restricting f to the line L, we get

$$f|_L = (a\lambda + b\mu)t + t^2\varphi(t)$$

for some function φ on t. By definition, $i_P(L,C) \geq 2 \iff a\lambda + b\mu = 0$, which gives us the unique line $L = a(x-x_0) + b(y-y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)$. □

Remark 3.3. It may happen that we have $i_P(L,C) \geq 3$ for the tangent line L. Such a point p is called an **inflection point**, or **flex**.

Example 3.4 (Flex on 'em)

Consider $f(x,y) = y - y^3 - x^3$ and $L = (y = 0)$. We have $(0,0)$ is a smooth point on $C = Z(f)$, with $\frac{\partial f}{\partial y} = 1$ and $\frac{\partial f}{\partial x} = 0$. Thus, $i_P(L,C) = f|_L$ is the multiplicity of 0 as a root of x^3, which is 3. Thus, the origin is a flex point.

3.1 Projective Plane Curves

We'll now begin working in $\mathbb{P}^2$ instead of $\mathbb{A}^2$. We already got a taste of the healthy benefits of $\mathbb{P}^2$ in Bezout's Theorem (1.2).

First, it would help if we explicitly define the projective plane. We'll assume $k = \overline{k}$ unless otherwise specified. The **projective plane** is defined as

$$\mathbb{P}^2 = (\mathbb{P}^2_k) := (\mathbb{A}^3 \setminus \{0\})/k^* = (\mathbb{A}^3 \setminus \{0\})/\sim,$$

where $(x_1,y_1,z_1) \sim (x_2,y_2,z_2) \iff \exists\,\lambda \in k^*$ such that $x_2 = \lambda x_1, y_2 = \lambda y_1, z_2 = \lambda z_1$. Colloquially, we are taking the space of lines through 0 in $\mathbb{A}^3$, which justifies why we are quotienting by scalar multiplication and removing the origin. We will use homogeneous coordinates $(x:y:z)$ to notate points in $\mathbb{P}^2$, so $(x_0:y_0:z_0)$ denotes the equivalence class of $(x_0,y_0,z_0) \in \mathbb{A}^3$.

We have a nice way of relating $\mathbb{A}^n$ with $\mathbb{P}^n$ (we'll stick with $n=2$ for now). There is an injection $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ sending $(x,y) \mapsto (x:y:1)$. (Quick sanity check: verify that this is injective!) Furthermore, we know how to specify the complement of the image of this injection, which gives us

$$\mathbb{P}^2 = \mathbb{A}^2_z \cup \{(x:y:0) \mid x,y \in k, (x,y) \neq (0,0)\}.$$

We denote the last set as L_∞ and call it the "**line at infinity**." This is what we meant in the first lecture when we "added a point at infinity" to every direction through the origin.

Note that our choice of fixing $z=1$ was arbitrary: we could define $\mathbb{A}^2_x = \{(1:y:z)\}$ and likewise for $\mathbb{A}^2_y$. We have $\mathbb{P}^2 = \mathbb{A}^2_x \cup \mathbb{A}^2_y \cup \mathbb{A}^2_z$. This will be a general phenomenon, which we'll see later: we can cover a projective variety by affine varieties. I know, I know, we haven't defined what a variety is yet, but you can think of a variety as a generalization of a plane algebraic curve, i.e. it is some zero locus.

Speaking of plane curves...

Definition 3.5 (Projective plane curve)**.** A **projective plane curve** is the zero locus of a nonconstant **homogeneous** polynomial $F \in k[X, Y, Z]$. (Homogeneous of deg d means all the monomials of F have deg d, i.e. $F(\lambda x, \lambda y, \lambda z) = \lambda^d \cdot F(x, y, z)$ for all $\lambda \in k$.

We just talked about multiplicity and intersection multiplicity for curves in affine space. How well do these notions port over in projective space?

This is why we want to consider only homogeneous polynomials. If F is homogeneous, then it makes sense to say that $(x : y : z)$ is a zero of F, because if $(x, y, z) \in \mathbb{A}^3$ is a zero of F, then so is $(\lambda x, \lambda y, \lambda z)$. So talking about zeroes is well-understood, and then we can proceed with talking about multiplicity of zeroes and intersection multiplicity.

We have a process of **homogenization**, which allows us to construct a homogeneous polynomial given some arbitrary polynomial. This allows us to take a curve in $\mathbb{A}^2$ and define its **projective closure** $\overline{C} \subseteq \mathbb{P}^2$. Suppose $C = Z(f(x, y) = 0) \subseteq \mathbb{A}^2$ and $\deg f = n$. Then, we can define

$$F(x, y, z) = z^n \cdot f(x/z, y/z).$$

This is homogeneous! For a simple example, we could take $f(x) = ax^2 + bx + c$ and consider $F(x, y) = y^2(a \cdot (x/y)^2 + b \cdot (x/y) + c)$. As said above, F defines a curve $\overline{C} = Z(F = 0) \subseteq \mathbb{P}^2$. Note that $F(x, y, 1) = f(x, y)$, so $C = \overline{C} \cap \mathbb{A}^2_z$.

We can also go the other direction. Suppose $F(x, y, z)$ is a homogeneous polynomial of deg n, which defines a curve $\overline{C} \subseteq \mathbb{P}^2$. Then, we can take the affine curves $\overline{C} \cap \mathbb{A}^2_x \subseteq \mathbb{A}^2$ defined by $F(1, y, z)$, and likewise for y and z.

3.2 Smooth Points and Tangent Lines in Projective Plane

We'll define smooth points and tangent lines for projective plane curves. Recall if $C = Z(f) \subseteq \mathbb{A}^2$, then a smooth point $p(x_0, y_0) \in C$ has a unique tangent line at p given by $\frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$.

Now let $C = Z(F) \subseteq \mathbb{P}^2$, where F is homogeneous of degree n. Let's first define smooth points. Take $p \in C$, and WLOG suppose $p \in \mathbb{A}^2_z$, so $p = (x_0 : y_0 : 1)$. Defining $f(x, y) = F(x, y, 1)$, we can now treat it as in the affine case: $\frac{\partial F}{\partial x}(x, y, 1) = \frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial F}{\partial y}(x, y, 1) = \frac{\partial f}{\partial y}(x, y)$. We now have an understanding of partial derivatives in projective plane, which allows us to define a point to be singular if the partials of F at p are all 0, i.e. $\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z}(p) = 0$. A point is nonsingular, or smooth, if it is not singular.

Exercise 3.6. Let $\operatorname{char} k = 0$. If F is a homogeneous polynomial of degree n, then we have *Euler's formula*

$$n \cdot F = X \cdot \frac{\partial F}{\partial x} + Y \cdot \frac{\partial F}{\partial y} + Z \cdot \frac{\partial F}{\partial z}.$$

Exercise 3.7. If we homogenize $\frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$ with respect to z and use Euler's formula, then the tangent line to C at p in $\mathbb{P}^2$ is given by

$$L : \frac{\partial F}{\partial x}(x_0, y_0, 1) \cdot (x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, 1) \cdot (y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, 1) \cdot (z - 1) = 0.$$

We'll now return to Bezout's Theorem, first giving a more preliminary version only involving projective plane curves.

Theorem 3.8 (Bezout's Theorem)
Let C, D be projective plane curves, C smooth, $C \not\subset D$. Then,

$$\sum_{P \in C \cap D} i_P(C, D) = \deg C \cdot \deg D.$$

We'll give examples and prove it for lines and conics next time.

4 02/01 - From Curves to the General Setting

Happy February! We'll open with looking at cubic curves. In the affine case, a cubic curve is defined by a degree 3 polynomial $f \in k[x, y]$. It turns out that a simple change of coordinates, we can do even better: we can write f as $y^2 = x^3 + ax^2 + bx + c$, and the curve defined by this equation is birational to $Z(f)$. We call this form the **Weierstrass normal form**.

We can improve from this even more. Sending $x \mapsto x - \frac{a}{3}$ gives the equation $y^2 = x^3 + px + q$ for some $p, q \in k$. We will refer to this resulting curve by C_{st}, where the "st" denotes that the curve is in "standard form." We now see when C_{st} is smooth:

Lemma 4.1
C_{st} is smooth if and only if $4p^3 + 27q^2 \neq 0$. (Here, assume $\operatorname{char} k \neq 0$.)

Proof. Note that $\frac{\partial f}{\partial y} = 2y = 0 \iff y = 0$. Let $g(x) = x^3 + px$. This means that if $(x_0, 0)$ is a singular point of the curve, then $g(x_0) = 0$ and $\frac{\partial g}{\partial x}(x_0) = 0$. Thus, g has a multiple root at x_0, which occurs iff the discriminant $\Delta = 4p^3 + 27q^2$ of g is 0. □

If $4p^3 + 27q^2 \neq 0$ (i.e. C_{st} is smooth), we call C_{st} an **elliptic curve**.

From your homework, you may observe that any irreducible singular cubic is rational (e.g. $y^2 = x^3, y^2 = x^3 + x^2$), but since elliptic curves are required to be smooth, we have the following:

Proposition 4.2
Elliptic curves are *not* rational.

Proof. Say $C_{\text{st}} : y^2 = x^3 + px + q$ were rational, i.e. birational to $\mathbb{A}^1$. Consider a map $\mathbb{A}^1 \xrightarrow{\varphi} \mathbb{A}^1 = k$. Then, we can write $\varphi(x) = \frac{P(x)}{Q(x)} = z \in k$ means $P(x) - Q(x) \cdot z = 0$. The only way to have exactly one solution in x for any z is if P, Q are linear polynomials. Letting $P(x) = ax + b$ and $Q(x) = cx + d$, we can solve for the fixed points via $ax + b = x(cx + d)$, which is just solving a quadratic. Thus, there are at most 2 fixed points.

In contrast, if we could find some map $C_{\text{st}} \to C_{\text{st}}$ which fixes more than two points, then we'd be done. But this is easy: consider the **involution** (i.e. a map that, when composed to itself, gives the identity) $(x, y) \mapsto (x, -y)$. □

4.1 Bezout's for a Line

We'll now prove Bezout's theorem in the special case of intersection between a line and a curve. This, we'll see, is simply a generalization of the Fundamental Theorem of Algebra in this case.

Theorem 4.3 (Baby version of Theorem 3.8)
Let $C = L \subseteq \mathbb{P}^2$ be a line and $D \subseteq \mathbb{P}^2$ a curve of degree d such that $L \not\subset D$, i.e. $|L \cap D| < \infty$. Then,

$$\sum_{P \in L \cap D} i_P(L, D) = d.$$

Proof. To make our lives easier from the start, we will change coordinates such that $L = (y = 0)$ and the line at infinity is $L_\infty = (z = 0)$. (This is an exercise – convince yourself that we can do this!) Further, L_∞ doesn't pass through the intersection points of L and D.

Considering the curve in affine space $\mathbb{A}^2_z$, D is given by some $f(x, y) = 0$ where $\deg f = d$. Write $f = f_d + f_{d-1} + \cdots + f_e$, where f_i are the terms of degree i. Note that $(1 : 0 : 0) \in L \cap L_\infty$, so it cannot lie on D. But then this means f_d must contain some term of the form $a_d x^d$ $(a_d \neq 0)$, otherwise when homogenized, y, z will appear in every term, giving $(1 : 0 : 0) \in D$.

This means that $i_P(L, D)$ is the multiplicity of P as a root of $f|_{L=(y=0)}$, which is simply a $\deg d$ polynomial in one variable x. Summing over all P in the intersection, we have $\sum i_P(L, D)$ is the sum of the multiplicities of all roots of $f|_K$ in x of degree d, which by the Fundamental Theorem of Algebra is simply d, as desired. □

We're going to see an application of Bezout's theorem, which some may be familiar with in an olympiad geometry setting! Projective geometry is very powerful.

Theorem 4.4 (Pascal's Theorem)
Let C be a conic with a hexagon $x_1, \dots, x_6$ inscribed in C. Denote ℓ_i as the line between x_{i-1} and x_i, and let $P = \ell_1 \cap \ell_4$, $Q = \ell_2 \cap \ell_5$, and $R = \ell_3 \cap \ell_6$. Then, P, Q, R are collinear.

One can prove this using some clever constructions or other more elementary methods, but this turns out to be a direct consequence of Bezout's.

Proof. For any $t \in k$, consider the cubic $C_t := (\ell_1\ell_3\ell_5 + t \cdot \ell_2\ell_4\ell_6 = 0)$. Note that x_i lies on the lines ℓ_i and ℓ_{i+1}, which means $x_1, \dots, x_6 \in C_t$ for any t. Now, fix some p distinct from the x_i's. Then, we may find some $s \in k$ such that $p \in C_s$. [justify this]

But then now, we have seven points $x_1, \dots, x_6, p \in C_s$. But Bezout's Theorem tells us that if $C \not\subset C_s$, then there can only be 6 intersection points, which means $C \subset C_s$. Given that $\deg C_s = 3$ and $\deg C = 2$, this means C_s decomposes into the conic C and a line L. The conclusion follows from the fact that $P, Q, R \notin C$, so they must lie on the line L. □

Now we will formally graduate from looking at baby examples and begin the general study.

4.2 Affine Varieties

Generalization of the affine plane:

Definition 4.5 (Affine space)**.** Let $k = \overline{k}$. The **affine space** over k is

$$\mathbb{A}^n = (\mathbb{A}^n_k) := \{(a_1, \dots, a_n) \mid a_i \in k\}.$$

Thus, we can understand a polynomial $f \in k[x_1, \dots, x_n]$ as a function $f : \mathbb{A}^n \to k$. This means that the zero locus of f will carve out a subset of $\mathbb{A}^n$, which we dealt with extensively when $n = 2$. We'll generalize, not just n, but the number of functions we're considering:

Definition 4.6. Let $S \subseteq k[x_1, \dots, x_n]$ be a subset. Then, the **zero set of S** is

$$Z(S) := \{x \in \mathbb{A}^n \mid f(x) = 0 \, \forall \, f \in S\}.$$

Definition 4.7 (Algebraic Set)**.** A subset $X \subseteq \mathbb{A}^n$ is called an **algebraic set** if $\exists \, S \subseteq k[x_1, \dots, x_n]$ such that $X = Z(S)$.

Example 4.8
In $\mathbb{A}^3$, the zero locus of $f = XY$ is the union of two planes $X = 0$ and $Y = 0$, and the zero locus of $f = X^2 + Y^2 - 1$ is a cylinder. Combining, the algebraic set

$X = Z(XY, X^2 + Y^2 - 1) \subseteq \mathbb{A}^3$ is the intersection of the two planes with the cylinder, which gives four disjoint lines.

So what we've laid out so far is not anything new – we knew that a cylinder could be given by $x^2+y^2-1 = 0$ since high school – but we're just making this connection between algebra and geometry explicit.

Remark 4.9. Some algebraic sets (not necessarily hypersurfaces) can be parameterized. For instance, the image of the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ sending $t \mapsto (t, t^2, t^3)$ carves out what we call the **"twisted cubic"** in $\mathbb{A}^3$. We can write out the twisted cubic explicitly as a zero locus: $C = Z(Y - X^2, Z - X^3)$.

Let's lay out our first algebra definition. Recall:

Definition 4.10 (Noetherian ring)**.** A ring R (commutative with unity) is **Noetherian** if every ideal in R is generated by finitely many elements. Equivalently, R has the *ascending chain condition* (ACC) on ideals, i.e. every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots \subseteq R$ stabilizes, i.e. $I_k = I_{k+1} = \cdots$ for some k.

Noetherian rings are fundamental because polynomial rings are one of the most fundamental examples of Noetherian rings, a consequence of the Hilbert Basis Theorem:

Theorem 4.11 (Hilbert Basis Theorem)

If R is Noetherian, then $R[X]$ is Noetherian.

The proof of this is nifty, and perhaps I'll present it in section.

5 02/06 - Building to Hilbert's Nullstellensatz

5.1 Zariski Topology

We continue our discussion of affine varieties from last time. Recall that we are now able to take the zero locus of not just a single function, but of a subset of $k[x_1, \ldots, x_n]$. Note that given subset $S \subseteq k[X_1, \ldots, X_n]$, we can construct the ideal $I(S) \leq k[X_1, \ldots, X_n]$ generated by elements of S. This is of interest to us because of the following:

Lemma 5.1

$Z(S) = Z(I(S))$.

Proof. Pretty straightforward. $\supseteq$ is clear; for $\subseteq$, if $x \in Z(S)$, then $f_i(x) = 0$ for all $f_i \in S$. Take $h = \sum g_i f_i \in I(S)$. Clearly, $h(x) = 0$, so $x \in Z(I(S))$. □

Thus, all algebraic sets are the zero set of some ideal $I \subseteq k[X_1, \dots, X_n]$. We now provide some basic facts about algebraic sets.

Lemma 5.2

Algebraic sets satisfy the following properties:

1. $\emptyset$ and $\mathbb{A}^n$ are algebraic sets.
2. If $S_1 \subseteq S_2$, then $Z(S_2) \subseteq Z(S_1)$.
3. If $(S_i)_{i \in I}$ is a family of subsets, then
$$\bigcap_{i \in I} Z(S_i) = Z\left(\bigcup_{i \in I} S_i\right).$$
4. If S_1, S_2 are subsets, then $Z(S_1) \cup Z(S_2) = Z(S_1 \cdot S_2)$.

Proof. Properties 1, 2, 3 are left as an exercise to the reader. For (4), we prove inclusion in both directions. For $\subseteq$, if $x \in Z(S_1) \cup Z(S_2)$, then $\forall\ f_1 \in S_1, f_2 \in S_2$, we have $f_1(x) = 0$ or $f_2(x) = 0$. But this means $f_1 f_2(x) = 0 \implies x \in Z(S_1 \cdot S_2)$. For the reverse direction, replace $x \in \dots$ with $x \notin \dots$ and $= 0$ with $\neq 0$ and the argument should follow. $\square$

Great. These properties seem pretty unassuming, but we can glean from the above lemma that (1) $\emptyset, \mathbb{A}^n$ are algebraic sets, (2) arbitrary intersections of algebraic sets are algebraic sets, and (3) finite unions of algebraic sets are algebraic sets! This means that

> There exists a **topology** on $\mathbb{A}^n$ where closed sets are the algebraic sets.

This topology is called the **Zariski topology**. This is the basis of all of algebraic geometry, and it's difficult to overstate its importance.

The Zariski topology is kinda bad as a topology though. First, it is much coarser than the standard topology: all closed subsets in the Zariski topology are closed in the classical topology. Furthermore, this topology is not Hausdorff: that is, any two open subsets will intersect. So the open subsets are a bit "weaker"/don't give us as much information, because they aren't as refined locally.

Example 5.3 (Zariski topology)

In $\mathbb{A}^1$, the closed sets of the Zariski topology are $\emptyset, \mathbb{A}^1$, and finite sets of points. This is because $k[x]$ is a PID, so any ideal is of the form (f), and $Z(f)$ for nonconstant f are simply the roots of f, of which there are only finitely many. In $\mathbb{A}^2$, the closed sets are $\emptyset$, $\mathbb{A}^2$, and plane curves union with finite sets of points.

5.2 Topological Properties

Naturally, we can define the subspace topology on any $X \subseteq \mathbb{A}^n$ given the Zariski topology on $\mathbb{A}^n$. We continue with more topological properties:

Definition 5.4 (Irreducible)**.** A topological space X is **irreducible** if there are no proper closed subsets X_1, X_2 such that $X = X_1 \cup X_2$. Otherwise, it is **reducible**.

This is a pretty useless notion in the usual topology; for instance, we can divide up $\mathbb{C}$ by the disk (with boundary) of radius 1 and the complement of the interior of said disk. But here, the notion of irreducibility is important. In fact, this directly gives us our main object of focus in this class.

Definition 5.5 (Affine Variety)**.** An irreducible affine algebraic set is called an **affine variety**.

Example 5.6 (Affine varieties)

If $f \in k[X_1, \ldots, X_n]$ is irreducible, then $Z(f) \subseteq \mathbb{A}^n$ is an affine variety. We call $Z(f)$ a **hypersurface** in $\mathbb{A}^n$. On the other hand, if we have something like $X = Z(xz, yz) \subseteq \mathbb{A}^3$, then we see that this is the union of the plane $(z = 0)$ and the line $(x = y = 0)$, so it is reducible. Finally, the twisted cubic curve given by the image of $\mathbb{A}^1 \to \mathbb{A}^3$, $t \mapsto (t, t^2, t^3)$ is an affine variety.

Another important property with respect to the Zariski topology:

Definition 5.7 (Noetherian)**.** A topological space is **Noetherian** if any descending class of closed subsets $X \supset X_1 \supset X_2 \supset \ldots$ is stationary (i.e. $\exists\, k_0$ such that $\forall\, k \geq k_0$, $X_k = X_{k+1}$).

We'll see that algebraic sets are Noetherian topological spaces, once we explicitly provide the useful correspondence between algebraic sets and ideals. But for now, we give a very nice characterization of Noetherian spaces:

Proposition 5.8

Every Noetherian space can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of irreducible closed subsets. If we assume $X_i \not\subset X_j$ for all $i \neq j$, then the decomposition is unique (up to reordering).

Proof. Suppose X is a Noetherian space not satisfying the proposition. Then, we must be able to write $X = X_1 \cup X_1'$ such that one of X_1, X_1' doesn't satisfy the proposition. WLOG let it be X_1. Then, we must be able to write $X_1 = X_2 \cup X_2'$ such that one of X_2, X_2' doesn't satisfy. WLOG let it be X_2. We continue ad infinitum to get a descending chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$, a contradiction to X being Noetherian.

For uniqueness, suppose for contradiction that $X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'$, where all X_i, X_j' are irreducible closed subsets. We induct on r. This means that $X_1 \subset X_1' \cup \cdots \cup X_s'$, so

$$X_1 = \bigcup_{j=1}^{s} (X_1 \cap X_j').$$

But X_1 is irreducible by hypothesis, so $\exists\, j$ such that $X_1 \subset X_j'$. By symmetry, $\exists\, i$ such that $X_j' \subset X_i$. But then $X_1 \subset X_j' \subset X_i$, which is only possible if $i = 1$ and $X_1 = X_j'$. Reindexing such that $j = 1$, we can throw out X_1, X_1' and look at

$$\begin{aligned} Z = \overline{X \setminus X_1} &= X_2 \cup \cdots \cup X_r \\ &= X_2' \cup \cdots \cup X_s'. \end{aligned}$$

The inductive hypothesis finishes the problem. □

Definition 5.9 (Dimension)**.** Let $X \neq \emptyset$ be an irreducible topological space. The **dimension** of X is the largest integer n such that there exists a chain of irreducible closed subsets

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X.$$

Although looks reasonable, this is actually a pretty hard definition to deal with tangibly:

Example 5.10

If $X = \mathbb{A}^n$, then we can consider $X = \mathbb{A}^n \supset \mathbb{A}^{n-1} \supset \mathbb{A}^{n-2} \supset \cdots \supset \mathbb{A}^1 \supset \mathbb{A}^0 \supset \emptyset$, so $\dim \mathbb{A}^n \geq n$. But "if there was any justice in this world, the dimension should be n." (quote from Popa) Can we prove this using only what we have? Seems like no, which is pretty unfortunate. It is true indeed that $\dim \mathbb{A}^n = n$, but we'll need some serious commutative algebra for this.

5.3 Between Algebraic Sets and Ideals

Now we start to make an explicit correspondence between algebra and geometry. This correspondence, naturally, is extremely powerful in algebraic geometry, and it allows us to do so many incredible things.

So first, we have a way of taking an ideal and making an algebraic set: this is $Z(\cdot)$. In particular, if $S \subseteq k[X_1, \ldots, X_n]$ is an ideal, then $Z(S)$ is the algebraic set of that ideal. We want something that goes the other way.

Definition 5.11. Let $X \subseteq \mathbb{A}^n$ be an arbitrary subset. The **ideal of X** is

$$I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \,\forall\, x \in X\}.$$

Restricting our attention to when X above is algebraic, There exists a one-to-one correspondence between $\{$algebraic sets in $\mathbb{A}^n\}$ and $\{$ideals in $k[x_1, \ldots, x_n]\}$.

We need to prove that this is actually a one-to-one correspondence, which we will do next time using Hilbert's Nullstellensatz.

Alongside this correspondence, we have two functions that take us between these two sets: $Z(\cdot)$ gives an ideal from an algebraic set, and $I(\cdot)$ goes the other direction. One naturally asks,

To what extent are these operations inverses to each other?

Let's look at some examples to make an informed guess.

Example 5.12

If $\mathcal{I} = (X^2) \subseteq k[X, Y]$, then $Z(I)$ is the y-axis, so $I(Z(\mathcal{I})) = (X)$. If $\mathcal{I} = (Y, Y^2 - X^2 - X^3) = (Y, X^2 + X^3) = (Y, X^2(X+1)) \subseteq k[X, Y]$, then by a similar reasoning to the (X^2) case, we get $I(Z(\mathcal{I})) = (Y, X(X+1))$.

So intuitively, what is happening is that we don't care about powers. This is the motivation for our following definition:

Definition 5.13. Let $I \leq R$ be an ideal (R commutative). The **radical of I** is the ideal

$$\sqrt{I} = \{f \in R \mid f^r \in I \text{ for some } r > 0\}.$$

Author's Note 5.14. I like the notation $\sqrt{I}$ for the radical, but some people use $\operatorname{Rad}(I)$ or $\operatorname{rad}(I)$. I will use interchangeably!

Exercise 5.15. Sanity check: $I \subseteq \sqrt{I}$.

The reason why we care about this is because this is exactly the condition we need in order for our correspondence between algebraic sets and ideals to hold: the vanishing ideal of an algebraic set is a radical ideal. In other words, if X is an algebraic set, then $I(X)$ is a radical ideal.

5.4 Continuing the Correspondence

Can we restrict our correspondence to affine varieties? (Recall these are just irreducible algebraic sets.) What kind of ideals do they correspond to?

Lemma 5.16
X is an affine variety if and only if $I(X)$ is a prime ideal.

Proof. Suppose X is reducible, i.e. $X = X_1 \cup X_2$ for some proper closed subsets X_1, X_2. Then, $I(X) \subsetneq I(X_1), I(X_2)$, which means $\exists\, f_1 \in I(X_1) \setminus I(X)$ and $f_2 \in I(X_2) \setminus I(X)$. But if $x \in X$, then either $f_1(x) = 0$ or $f_2(x) = 0$, which means $(f_1 \cdot f_2)(x) = 0 \implies f_1 f_2 \in I(X)$, so $I(X)$ is not prime.

Now suppose $I(X)$ is not prime. Then, there exists $f, g \notin I(X)$ such that $f \cdot g \in I(X)$. Then, we have $X \subseteq Z(f \cdot g) = Z(f) \cup Z(g)$, so we can write

$$X = (X \cap Z(f)) \cup (X \cap Z(g)).$$

But as $f, g \notin I(X) \implies X \not\subseteq Z(f), Z(g)$, we have just written X as a union of two proper closed subsets, so X is reducible. □

So this is a really nice way of determining whether an algebraic set is a variety, because we can easily check if an ideal is prime: we just consider its quotient and check if it is an integral domain.

5.5 Hilbert's Nullstellensatz (assuming Weak)

Now we present a theorem which will be the focus of next lecture. This is very much in the realm of commutative algebra, thus next time will be a very algebra-heavy lecture.

Theorem 5.17 (Hilbert's Nullstellensatz, Variant I/"Weak Nullstellensatz")
Let $k = \overline{k}$ and $R = k[X_1, \ldots, X_n]$. The maximal ideals in this ring are precisely those of the form $\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n)$, $a_i \in k$.

Corollary 5.18
Every ideal $I \subsetneq k[X_1, \ldots, X_n]$ has $Z(I) \neq \emptyset$.

Proof. This follows because every ideal I is contained in some maximal $\mathfrak{m}$, which is of the form $(X_1 - a_1, \ldots, X_n - a_n)$ by Weak Nullstellensatz. Thus, $\{(a_1, \ldots, a_n)\} = Z(\mathfrak{m}) \subset Z(I)$, as desired. □

Remark 5.19. Note that this miserably fails in non-algebraically closed fields. For instance, $X^2 + 1 \subseteq \mathbb{R}[X]$ generates a maximal ideal, but it has no roots.

But assuming this version of Nullstellensatz (German for "root theorem"), we can prove our claimed correspondence between algebraic sets and radical ideals.

Theorem 5.20 (Hilbert's Nullstellensatz, Variant II)
If X is an algebraic set in $\mathbb{A}^n$, then $Z(I(X)) = X$. If $I \subseteq k[X_1, \ldots, X_n]$ is an ideal, then $I(Z(I)) = \sqrt{I}$.

Proof. We start with the first statement. $\supseteq$ is clear (sanity check!), so we will show $\subseteq$. Let $X = Z(\mathcal{I})$; then, $\mathcal{I} \subseteq I(Z(\mathcal{I})) = I(X) \implies Z(I(X)) \subseteq Z(\mathcal{I})$, as desired.

The second statement is harder. One direction is clear: convince yourself that $\sqrt{I} \subseteq I(Z(I))$ follows from definition. There are many ways, some very involved/heavy on commutative algebra, to prove the reverse inclusion, but we will execute "Rabinowitsch Trick," where we will consider one more variable and consider an ideal in the larger ring. Explicitly, let $I = (f_1, \ldots, f_r)$, and take $g \in I(Z(I)) \subseteq k[X_1, \ldots, X_n]$. We can construct a new ideal $\mathcal{J} = (f_1, \ldots, f_r, X_{n+1} \cdot g - 1) \subseteq k[X_1, \ldots, X_n, X_{n+1}]$.

Take $p \in Z(\mathcal{J})$, so $f_1(p) = \cdots = f_r(p) = 0$ by definition. But this means $g(p) = 0$ since $g \in I(Z(f_1, \ldots, f_r))$. At the same time, $p \in Z(\mathcal{J})$ means $(X_{n+1} \cdot g - 1)(p) = 0$, which is only possible if $-1 = 0$, a contradiction. Thus, $Z(\mathcal{J}) = \emptyset$. Now we whip out our machete: by Weak Nullstellensatz, this is equivalent to saying $\mathcal{J} = k[X_1, \ldots, X_{n+1}]$, so $\exists\, h_1, \ldots, h_{r+1} \in k[X_1, \ldots, X_{n+1}]$ such that

$$\sum_{i=1}^{r} h_i \cdot f_i + h_{r+1}(X_{n+1} \cdot g - 1) = 1.$$

Although this may look complicated, we now have an identity in terms of the variables $X_1, \ldots, X_{n+1}$. In particular, this identity must continue to hold if we set $X_{n+1} = \frac{1}{g(X_1, \ldots, X_n)}$. This gives us

$$\sum_{i=1}^{r} h_i \left(X_1, \ldots, X_n, \frac{1}{g(X_1, \ldots, X_n)} \right) \cdot f_i(X_1, \ldots, X_n) = 1.$$

We can multiply by a sufficiently large power of g (say g^N) to clear denominators in the left hand side, from which we get, for some $h_i' \in k[X_1, \ldots, X_n]$, $\sum_{i=1}^{r} h_i' \cdot f_i = g^N \implies g^N \in I$, as desired. $\square$

This validates our correspondence, which we now lay out in full.

1. We have a bijection {algebraic sets in affine $\mathbb{A}^n$} $\leftrightarrow$ {radical ideals in $k[x_1, \ldots, x_n]$}, where the arrows are given by $\xrightarrow{I(\cdot)}$ and $\xleftarrow{Z(\cdot)}$.
2. This restricts to a 1-1 correspondence between {affine varieties in $\mathbb{A}^n$} and {prime ideals in $k[x_1, \ldots, x_n]$}.
3. This further restricts to a 1-1 correspondence between {points in $\mathbb{A}^n$} and {maximal ideals in $k[x_1, \ldots, x_n]$}.

We now explicitly define something we've encountered many times in the homework.

Definition 5.21 (Coordinate Ring)**.** Let $X \subsetneq \mathbb{A}^n$ be an affine variety. Then,

$$A(X) := k[X_1, \ldots, X_n]/I(X)$$

is the **affine coordinate algebra** of X.

Remark 5.22. We can consider the coordinate algebra of an algebraic set in general, but note that the ring will no longer be integral since $I(X)$ need not be prime.

Elaborating more on the above, if $A(X)$ is not an integral domain, then (0) is no longer a prime ideal. To see what instead the minimal primes are, we can consider the decomposition $X = X_1 \cup \cdots \cup X_r$ into irreducible components. Since each X_i is irreducible, we have $I(X_i) = \mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i \subset k[x_1, \ldots, x_n]$. Then, we have a **primary decomposition** $I(X) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$.[1] The $\mathfrak{p}_i$'s are the minimal primes in $k[x_1, \ldots, x_n]$ that contain $I(X)$, hence are the minimal primes in $A(X)$.

5.6 Affine Theory of Dimension

Dimension Theory is extremely expansive in commutative algebra and algebraic geometry, enough so that we could talk about this for a whole semester, so we have to choose our battles here. To prove Nullstellensatz in full, we will forego proving lots of results here and instead provide the information necessary to apply these results in our class.

If you've studied some commutative algebra before, though, this will be familiar notions to you!

Definition 5.23 (Krull Dimension)**.** Let R be a ring. The **(Krull) dimension** of R is

$$\dim R := \sup_{\mathfrak{p} \text{ prime}} \{\operatorname{ht} \mathfrak{p}\},$$

where $\operatorname{ht} \mathfrak{p}$ is the sup of the length k over any chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k = \mathfrak{p}$.

Remark 5.24. Unfortunately, even for Noetherian rings, we can have $\dim R = \infty$, but these edge cases are largely pathological, so we're chillin'.

Exercise 5.25. $\dim X = \dim A(X)$. (See how the definitions correspond to each other!)

Now we state our main theorem of Dimension Theory.

[1] This is a term from commutative algebra. Look it up to learn more, but here, think of it as factorization into prime ideals.

Theorem 5.26 (Main Theorem of Dimension Theory)
Let k be a field and B be an integral, finitely generated k-algebra. Then,

1. $\dim B = \operatorname{trdeg}_k Q(B)$, where $Q(B)$ is the field of fractions of B.
2. For all prime ideals $\mathfrak{p} \subseteq B$, $\operatorname{ht} \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$.

Although we won't provide a proof, we discuss a little bit of its application to see what's going on.

Example 5.27
If $B = k[x_1, \ldots, x_n]$, then $Q(B) = k(x_1, \ldots, x_n)$, which is perhaps the simplest example of a purely transcendental extension. This means $\dim B = \operatorname{trdeg}_k k(x_1, \ldots, x_n) = n$, yay!

More generally, if we consider $A(X) = k[x_1, \ldots, x_n]/I(X)$, then we can think of $A(X) = k[\overline{x_1}, \ldots, \overline{x_n}]$, with these $\overline{x_i}$'s having algebraic relations dictated by $I(X)$. In particular, up to some x_r, the elements $x_1, \ldots, x_r$ will be algebraically independent, so we can think of the tower of extensions

$$k \subseteq k(x_1, \ldots, x_r) \subseteq k(x_1, \ldots, x_r)(x_{r+1}, \ldots, x_n) = k(x_1, \ldots, x_n),$$

where the first extension is purely transcendental and the second is algebraic. This gives $\dim A(X) = r$.

6 02/13 - Commutative Algebra

Commutative algebra time.

Theorem 6.1 (Krull's Principal Ideal Theorem)
Let R be a Noetherian ring, $f \in R$ such that f is neither a zero divisor nor a unit. Then, there exists a minimal prime $\mathfrak{p}$ over (f) such that $\operatorname{ht} \mathfrak{p} = 1$.

This allows us to make the following precise: if we're considering the coordinate ring of a curve C carved out by a single $f(x_1, \ldots, x_n)$, then we expect the dimension of the curve to be $n - 1$. This is true from the "Main Theorem of Dimension Theory" (5.26), as we have $\operatorname{ht} \mathfrak{p} + \dim A[C]/\mathfrak{p} = \dim A[C]$. We make this both more general and more precise by interpretating this algebraic statement geometrically:

Theorem 6.2 (Geometric Version of Theorem 6.1)
If $X \subseteq \mathbb{A}^n$ is an algebraic set and $f \in k[x_1, \dots, x_n]$ such that $Z(f)$ does not contain any component of X but intersects with X, then there exists a component of $X \cap Z(f)$ with dimension equal to $\dim X - 1$.

The following corollary is what I was trying to talk about with the dimension of a curve being $n - 1$.

Corollary 6.3
Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then, $\dim X = n - 1 \iff X = Z(f)$ for some irreducible polynomial $f \in k[x_1, \dots, x_n]$.

Proof. This is equivalent to saying a prime ideal $\mathfrak{p} \subseteq k[x_1, \dots, x_n]$ has $\operatorname{ht} \mathfrak{p} = 1$ iff $\mathfrak{p} = (f)$ for some irreducible f. We first prove the forward direction: take some $0 \neq f \in \mathfrak{p}$. Since $k[x_1, \dots, x_n]$ is a UFD, we can factor f into irreducibles. But $f \in \mathfrak{p}$ means that at least one irreducible factor belongs to $\mathfrak{p}$, so we may assume f itself is irreducible. This means $0 \subsetneq (f) \subseteq \mathfrak{p}$, and $\operatorname{ht} \mathfrak{p} = 1$ immediately gives $(f) = \mathfrak{p}$. The reverse direction immediately follows from Krull's Principal Ideal Theorem. □

Remark 6.4. This nice correspondence completely breaks down if we're considering varieties that decrease the dimension by more than 1. The only situation where this analogy carries over is in complete intersection rings.

6.1 Finiteness and Integrality Conditions

May be slightly boring, but very necessary (otherwise we wouldn't be spending so much time on it!).

Definition 6.5 (Types of finitely generated)**.** Let $R \subseteq S$ be two rings. Then,

1. S is **finitely generated** over R (**module-finite** over R) if $\exists\, s_1 \dots, s_n \in S$ which generates S as an R-module, i.e. $\forall\, s \in S$, $\exists\, r_i \in R$ such that $s = r_1 s_1 + \cdots + r_n s_n$.
2. S is **finitely generated as an R-algebra** (**ring-finite** over R) if $\exists\, s_1, \dots, s_n \in S$ such that $S = R[s_1, \dots, s_n]$.

If S is ring-finite over R, then there exists a surjection

$$\begin{aligned} R[X_1, \dots, X_n] &\twoheadrightarrow R[s_1, \dots, s_n] = S \\ R &\mapsto R \\ X_i &\mapsto s_i. \end{aligned}$$

The kernel of this map dictates the relations between these generators s_i. Thus, an alternate way of talking about ring-finiteness is to say that there exists a surjection from the polynomial ring to S.

Exercise 6.6. (Sanity check) Module-finite implies ring-finite.

These finiteness conditions behave well under composition:

Proposition 6.7

Module-finiteness and ring-finiteness are preserved under compositions of inclusions, i.e. if S is module/ring finite over R and T is module/ring finite over S, then T is module/ring finite over R.

Proof. On the homework, mwahahaha. □

Definition 6.8 (Integral)**.** If $R \subseteq S$ and $s \in S$, we say s is **integral** over R if there exists a *monic* polynomial $f \in R[x]$ such that $f(s) = 0$. We say S is **integral** over R if every element of S is an integral element.

Example 6.9

$\mathbb{Q}[\sqrt{2}]$ is integral over $\mathbb{Q}$, as $\sqrt{2}$ satisfies $x^2 - 2 = 0$. An extension like $\mathbb{Q}[\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \dots]$ is also an integral extension over $\mathbb{Q}$, but it is not ring-finite. On the other hand, $\mathbb{Q}[\pi]$ is not integral over $\mathbb{Q}$.

6.2 Relating the Two

How can we relate these two notions (finiteness and integrality) together? How can we characterize integral elements?

Proposition 6.10

Let $R \subseteq S$, S an integral domain, $s \in S$. Then the following are equivalent:

1. s is integral over R,
2. $R[s]$ is module-finite over R
3. $\exists\, R[s] \subseteq R' \subseteq S$ subring that is module-finite over R.

Proof. (1) $\implies$ (2): We know there exists some $f = x^n + r_{n-1}x^{n-1} + \cdots + r_0 \in R[x]$ such that $f(s) = 0$. Thus, we can write $s^n = -r_{n-1}s^{n-1} - \cdots - r_1 s - r_0$, which means s^n

(and thus all higher powers of s) are contained in the module-finite extension generated by $1, s, \ldots, s^{n-1}$. The conclusion follows.

(2) $\implies$ (3): Obvious, take $R' = R[s]$.

(3) $\implies$ (1): Suppose R' is generated over R by $v_1, \ldots, v_n \in R'$. Then, we can express $s \cdot v_i = \sum_{j=1}^{n} a_{ij} \cdot v_j$. Expressing this in terms of matrices, let $A = (a_{ij})$ and v the column vector consisting of $v_1, \ldots, v_n$. Then, we have $A \cdot v = (s \cdot I_n) \cdot v \implies (A - sI_n) \cdot v = 0$. Since $v \neq 0$, this means $\det(A - sI_n)$. The conclusion follows from the observation that the determinant is a monic polynomial in s with coefficients in R, which implies s is integral over R, as desired. □

This result is really nice, as we observe from the following consequences. The first nice consequence of the above proposition is that the set of integral elements has a subring structure:

Corollary 6.11

The set of elements in S that are integral over R is a subring $\overline{R} \subseteq S$ containing R. We call $\overline{R}$ the **integral closure** of R in S.

You may have observed this phenomenon when $R \subseteq S$ is a field extension from some field theory class. This is a generalization.

Proof. Let $a, b \in S$ be integral over R. By the above proposition (Prop 6.10), $R[a]$ is module-finite over R. Since b is integral over R, it is clearly integral over $R[a]$. Using Proposition 6.10 again, we see that $R[a, b]$ is module-finite over $R[a]$. Since finiteness is preserved under compositions (Proposition 6.7), we have $R[a, b]$ is module-finite over R.

This gives us what we need. We want to show that $a + b$ and $a \cdot b$ are both integral over R. But both of these lie in $R[a, b]$, so take some arbitrary $s \in R[a, b]$. Then, we have $R[s] \subseteq R[a, b] \subseteq S$, and the (3) $\implies$ (1) implication of Proposition 6.10 tells us that s is integral over R. □

Corollary 6.12

Suppose S is ring-finite over R. Then, S is module-finite over R iff S is integral over R.

Proof. We first prove the forward direction. Suppose $s \in S$, so $R[s] \subseteq S$. Using the last implication of Proposition 6.10, this immediately gives us s is integral.

For the reverse direction, suppose $S = R[s_1, \ldots, s_n]$ where each s_i is integral over R. We proceed by induction on n. For $n = 1$, it is clear that $R \subseteq R[s_1]$ is module-finite by the first implication of Proposition 6.10. Now suppose $R[s_1, \ldots, s_k]$ is module-finite over R. Then, s_{k+1} is integral over R means it is integral over $R[s_1, \ldots, s_k]$, which means $R[s_1, \ldots, s_{k+1}]$ is module-finite over $R[s_1, \ldots, s_k]$ by the first implication of the proposition again. Using our inductive hypothesis and Proposition 6.7 (finiteness preserved under composition), we have $R[s_1, \ldots, s_{k+1}]$ is module-finite over R, as desired. □

6.3 Applying to Fields

We'll port over everything now to the specific case where R, S from above are actually fields. We have slightly different terminology for the fields case, but bear with me. Let $K \subseteq L$ be a field extension and $s_1, \ldots, s_n \in L$. Observe that $K(s_1, \ldots, s_n)$, the field of fractions of $K[s_1, \ldots, s_n]$, is the subfield of L generated by $K, s_1, \ldots, s_n$.

Definition 6.13.

1. L is a **finitely generated field extension** of K if $L = K(s_1, \ldots, s_n)$ for some $s_1, \ldots, s_n \in L$.
2. L is an **algebraic extension** over K if all elements in L are algebraic over K. The set of all algebraic elements over K forms a subfield $K \subseteq \overline{K} \subseteq L$, called the **algebraic closure** of K in L.

6.4 Proving Weak Nullstellensatz

To remind you of why we're doing all of this in the first place, we proved Hilbert's Nullstellensatz (Theorem 5.20) assuming the Weak Nullstellensatz (Theorem 5.17), but we never got around to proving the Weak version. We now state a very powerful statement, which we'll prove next time, that allows us to provide a proof.

Theorem 6.14

Let $K \subseteq L$ be a field extension. If L is ring-finite over K, then L is module-finite (i.e. algebraic) over K.

Proof. (of Weak Nullstellensatz) Let $k = \overline{k}$, $R = k[X_1, \ldots, X_n]$, and $\mathfrak{m} \subset R$ a maximal ideal. Then, we have a map $k \hookrightarrow R/\mathfrak{m} = L$, where $R/\mathfrak{m} = k[\overline{X_1}, \ldots, \overline{X_n}]$ is ring-finite over k. By the above theorem (6.14), L is algebraic over k. But k is algebraically closed, so this forces $k = L$, which means $\overline{X_i} = \overline{a_i} \in R/\mathfrak{m}$ for some $a_i \in k$. This implies $(X_1 - a_1, \ldots, X_n - a_n) \subseteq \mathfrak{m}$, but the former is already a maximal ideal, so equality follows. □

7 02/15 - Wrapping up Commutative Algebra (for now)

7.1 Completing Nullstellensatz Proof

Recall Theorem 6.14 above. This is a crucial result about fields: there is no distinguishing between ring-finite and module-finite for fields! Using it, we proved Weak Nullstellensatz, which gets us closer to completing our proof of Hilbert's Nullstellensatz (recall we first assumed Weak to prove Hilbert, then we proved Weak assuming the above theorem at the end of last class).

Proposition 7.1

Suppose k a field and X an indeterminate variable.

1. $k(X)$ is a finitely generated field extension that is *not* ring finite over k.
2. $k[X] = \overline{k[X]}$ inside $k(X)$.

Proof. The first statement is left as an exercise. (`Hint:` clear denominators.) For the second, suppose $s = P/Q \in k(X)$ is integral over $k[X]$, where $P, Q \in k[X]$ are relatively prime. s being integral means $\exists\, f_i \in k[X]$ such that $s^n + f_{n-1}s^{n-1} + \cdots + f_0 = 0$. Substituting $s = P/Q$ and clearing denominators, we have

$$P^n + f_{n-1}P^{n-1}Q + \cdots + f_0 Q^n = 0.$$

From here, we can deduce $Q \mid P^n$, but since we assumed P, Q are coprime, this is only possible if Q is constant, so $s \in k[X]$. □

Now we will prove Theorem 6.14.

Proof. Since L is ring-finite over K, we may write $L = K[s_1, \ldots, s_n]$ for $s_i \in L$. We proceed by induction on n. If $n = 1$, then we have a map $\varphi : K[x] \twoheadrightarrow K[s_1] = L$ where K maps identically and $x \mapsto s_1$. By First Isomorphism Theorem, we have $L = K[x]/I$. But if L is a field, this means I is maximal, so $I = (f)$ for some monic irreducible polynomial f. But this means that $\varphi(f(x)) = f(s_1) = 0$ in L, so s_1 is indeed algebraic over K.

Now assume the statement is true for $n - 1$ elements. Applying this to the base field $K(s_1)$, we have that $K(s_1)[s_2, \ldots, s_n]$ is an algebraic extension of $K(s_1)$ by the inductive hypothesis. If s_1 is also algebraic over K, then we are done. Otherwise, s_1 is transcendental, so $K(s_1) \cong K(x)$.

Since each of the s_i are algebraic ($2 \le i \le n$), we have $f_{ij} \in K(s_1)$ such that for all $2 \le i \le n$,

$$s_i^{n_i} + f_{i,n_i-1} \cdot s_i^{n_i-1} + \cdots + f_{i,0} = 0.$$

We know that there exists some $f \in K[s_1]$ such that multiplying by f will clear all denominators in each f_{ij} (for an explicit construction, just let $f_{ij} = g_{ij}/h_{ij}$ and let $f = \prod_{i,j} h_{ij}$). For each i, we will multiply by f^{n_i} to get

$$(fs_i)^{n_i} + f \cdot f_{i,n_i-1}(fs_i)^{n_i-1} + \cdots + f^{n_i} f_{i,0} = 0,$$

where all coefficients of (fs_i) are now in $K[s_1]$. Thus, $f \cdot s_i$ are all integral over $K[s_1]$.

Let $L = K[s_1, \ldots, s_n]$. If $t \in L$, then there must exist some $N > 0$ such that $f^N \cdot t \in K[s_1][fs_2, \ldots, fs_n]$, meaning $f^N \cdot t$ is integral over $K[s_1]$. Now take any $g \in K[s_1]$ relatively prime to f, and consider $t = g^{-1} \in K(s_1)$. From what we just said, we have f^N/g is integral over $K[s_1]$ for some large enough $N > 0$. By Proposition 7.1, $K[s_1]$ is integrally closed in $K(s_1)$, so $f^N/g \in K[s_1]$, a contradiction since we assumed f, g are relatively prime. □

Although the proof may look a bit lengthy and involved, most of the things we did were just algebraic tricks where we just clear denominators and work with polynomials. The crux of the proof is our application of Proposition 7.1 at the end. Clutch stuff.

7.2 Regular Functions

Whenever we study a new mathematical object, it is just as, if not more, important to study the maps between these objects.

Suppose $X \subseteq \mathbb{A}^n$ is an affine variety. We already defined the affine coordinate ring $A(X) = k[X_1, \ldots, X_n]/I(X) = \{\text{polynomials } f : X \to k\}$. We think of these as "regular functions on X," although we have yet to define what regular means. Let's do this now:

Definition 7.2. Let $X \subseteq \mathbb{A}^n$ affine variety and $x \in X$. Define the **local ring of x in X**

$$\mathcal{O}_{X,x} = \left\{ \varphi = \frac{f}{g} \mid f, g \in A(X), g(x) \neq 0 \right\}.$$

We can think of $\mathcal{O}_{x,x}$ as the subring of $K(X)$ the **function field of X**, which we can also think of $\text{Frac}(A(X))$ the field of fractions of $A(X)$, where the elements are defined (**regular**) at x. (This is the $g(x) \neq 0$ condition.)

Remark 7.3. The name "local ring" may ring (no pun intended) a bell from commutative algebra, if you've done that stuff before. We'll see shortly that it is indeed a local ring. (For the ambitious: can you guess what the unique maximal ideal is?)

Note that $\mathcal{O}_{X,x}$ is carrying extremely local – in fact, point-specific – information: it contains all rational functions that are regular at a specific point x. We can also generalize this to local neighborhoods: if $U \subseteq X$ is open, then we can define

$$\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_{X,x} \subseteq K(X).$$

This is the ring of rational functions that are regular on all of U, i.e. for every $x \in U$, every $\varphi \in \mathcal{O}_X(U)$ is defined (regular) at x.

Remark 7.4. If the notation $\mathcal{O}_{X,x}$ and $\mathcal{O}_X(U)$ seems a bit weird/seems to come out of nowhere, don't worry. Because we're only staying in the classical realm of algebraic geometry, we're sweeping a lot of subtleties/bigger picture stuff under the rug. If you want to look more into it, then what we're really talking about here is **sheaves**, which associates to every open subset U the regular functions defined on U. We call $\mathcal{O}_{X,x}$ the **stalk** of the sheaf at x, and this is the ring of germs of functions at x. (Lots of new words, but the stalk just carries very local information.)

Example 7.5 (Regular functions expressed differently)

Important: just because some $\varphi \in \mathcal{O}_X(U)$ is regular at every point in U doesn't mean that it is "expressed the same" at every point. Here's what I mean: consider the open subset

$$U = \{(x, y, z, t) \in X \mid y \neq 0 \text{ or } t \neq 0\} \subseteq X = (xt - yz = 0) \subseteq \mathbb{A}^4.$$

Consider the rational function $\varphi = \frac{x}{y}$. This is regular on all of U except for when $y = 0$, but luckily in our field of fractions, $\frac{x}{y} = \frac{z}{t}$, and $\frac{z}{t}$ is regular for $y = 0$, so φ is regular on all of U even though we don't have a consistent choice of writing φ at every point.

7.3 Local Ring Aside

Actually, Popa is talking about the terminology "local ring" so I'll insert this aside here:

Definition 7.6 (Local Ring)**.** R is a **local ring** if one, and hence both, of the following is true:

1. There exists only one maximal ideal $\mathfrak{m} \leq R$.
2. There exists a maximal ideal $\mathfrak{m} \leq R$ such that all $x \in R \setminus \mathfrak{m}$ is invertible.

Exercise 7.7. (Algebra exercise!) Show that these two conditions are equivalent.

Example 7.8

$k[X_1, \ldots, X_n]$ is not a local ring, since any ideal of the form $(X_1 - a_1, \ldots, X_n - a_n)$ are maximal. On the other hand, $k[[X_1, \ldots, X_n]]$ is a local ring! Can you find the unique maximal ideal? (Answer: $(X_1, \ldots, X_n)$.)

Now why is $\mathcal{O}_{X,x}$ a local ring? Consider the evaluation map

$$\begin{aligned} \mathrm{ev}_x : \mathcal{O}_{X,x} &\twoheadrightarrow k \\ \varphi &\mapsto \varphi(x) = f(x)/g(x). \end{aligned}$$

This is clearly surjective, and it is well-defined. Consider the ideal $\mathfrak{m}_x = \ker(\mathrm{ev}_x) = \{\varphi = f/g \mid f(x) = 0, g(x) \neq 0\}$. By First Isomorphism Theorem, $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$, which is a field, so $\mathfrak{m}_x$ is maximal! Furthermore, one can use the second condition of local rings to show that anything outside of $\mathfrak{m}_x$ is invertible, so it is the unique maximal ideal.

We could continue for a really long time with this aside, but hopefully this gives you a taste of just how geometric algebra is. Local rings are called that because they have this explicit geometric interpretation of rings that carry, well, local data.

Even better, although these local rings carry just local data a priori, it turns out that lots of global information (e.g. information about all of X) can be recovered just from $\mathcal{O}_{X,x}$. And this is nice because $\mathcal{O}_{X,x}$ is a really well-behaved ring: not only is it local, but also...

Exercise 7.9. $\mathcal{O}_{X,x}$ is a Noetherian ring.

This is left as an Exercise to the Reader (hehe), but if you know what localization is, consider the localization of $A(X)$ by the ideal of regular functions vanishing at x. The exercise follows from the fact that any localization of a Noetherian ring is still Noetherian (verify this!).

7.4 Poles

Points where a rational function is not defined are important to study in many areas of math (complex analysis, algebraic geometry, etc), so we give them a name.

Definition 7.10 (Poles)**.** $x \in X$ is a **pole** of $\varphi \in K(X)$ if φ is not regular at x.

Example 7.11 (Pole)
Consider $X = (xy - z^2 = 0) \subseteq \mathbb{A}^3$ and the rational function $\varphi = \frac{x}{z} = \frac{z}{y}$. Then, $(0, 0, 0)$ is a pole for φ.

Lemma 7.12
The set of poles of a rational function on X is a closed subset of X.

Proof. Let $X \subset \mathbb{A}^n$ affine and $\varphi \in K(X)$. Consider the ideal of $A(X)$

$$I_\varphi = \{h \in A(X) \mid h \cdot \varphi \in A(X)\}.$$

I claim that the poles of φ are exactly $Z(I_\varphi)$. This follows because x is *not* a pole iff (by definition) there exists an expression $\varphi = f/g$ such that $g(x) \neq 0$. This means $g \in I_\varphi$, and so $g(x) \neq 0 \iff x \notin Z(I_\varphi)$. □

I'm not sure exactly how the following remark came up (maybe it was in response to a question), but it is an important one so I want to make it explicit.

Exercise 7.13. There exists a one-to-one correspondence between ideals in R/I and ideals in R containing I.

This is a purely algebraic fact, and it is not difficult to show, but it is an extremely important correspondence. On the note of it being "easy to see":

Popa: We have this correspondence... I don't know what it's called.

Eliot: I've always heard of it as the Correspondence Theorem.

Popa: Well, it's more of a Correspondence Observation.

Lemma 7.14
$\mathcal{O}_X(X) := \bigcap_{x \in X} \mathcal{O}_{X,x} = A(X)$.

Proof. Reverse inclusion is clear. For $\subseteq$, suppose $\varphi \in \bigcap \mathcal{O}_{X,x}$, i.e. φ is no poles. By the proof of the above lemma, this means $Z(I_\varphi) = \emptyset$, and from Weak Nullstellensatz we have $I_\varphi = A(X)$. Thus, $1 \in I_\varphi$, so $\varphi \in A(X)$. □

8 02/22 - Morphisms

Missed the beginning of class, but we talked more about $\mathcal{O}_{X,x}$ being a Noetherian local ring, including definitions of local and such. Included a bit of the discussion in the above section.

Author's Note 8.1. Happy birthday Eliot!!

8.1 Regular Maps

These are the **morphisms** between algebraic sets. Last time, we said the set of regular maps at a point x forms a ring, which we called the local ring at x. Let's more formally define a regular map:

Definition 8.2 (Regular Map)**.** Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A function $f : X \to Y$ is a **regular map** (**morphism**) if $\exists\, P_1, \ldots, P_m \in k[X_1, \ldots, X_n]$ such that $f(x) = (P_1(x), \ldots, P_m(x))$ for all $x \in X$. This is summarized by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}^n & \xrightarrow[(P_1,\ldots,P_m)]{} & \mathbb{A}^m \end{array}$$

Example 8.3

Consider the x-axis $X = \mathbb{A}^1$ and the parabola $Y = Z(y - x^2)$. We have a really natural map from X to Y where we "project up," i.e. any point x maps to (x, x^2). So we have a map $f : \mathbb{A}^1 \to \mathbb{A}^2$ where $t \mapsto (t, t^2)$, and this induces a map onto Y. Note that we could also go the other direction: take the projection map $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ onto the x-axis, and this induces a morphism from $Y \to X$.

Definition 8.4 (Isomorphism)**.** A regular map such that it has an inverse which is also a regular map is called an **isomorphism**.

Example 8.5 (Isomorphism)

Or rather, non-isomorphism. Let $X = \mathbb{A}^1$ and $Y = Z(y^2 - x^3)$. Note that $\mathbb{A}^1$ is smooth but Y has a cusp, so geometrically we can guess that they aren't isomorphic, at least not at the cusp $(x, y) = (0, 0)$. Indeed, this is exactly the case: if we take the function $f : \mathbb{A}^1 \to \mathbb{A}^2$ where $t \mapsto (t^2, t^3)$, then the image is Y, so $f : X \to Y$ is a bijective morphism. However, it is does not have an inverse, because its inverse $g : Y \to X$ mapping $(x, y) \mapsto y/x$ is *not defined at* 0. We will see this situation appear again and again, so be wary of it.

Let $f : X \to Y$ be a morphism. We can then define the **pullback map** on the coordinate rings. f induces a map

$$f^* : A(Y) \to A(X)$$
$$g \mapsto g \circ f.$$

Exercise 8.6. Check that this is "well-behaved," i.e. it is a k-algebra homomorphism. Further check that it respects composition, i.e. $(f \circ h)^* = h^* \circ f^*$.

Remark 8.7. Here, it is useful to adopt the perspective that $A(X) = \mathcal{O}_X(X)$, the ring of regular maps on all of X. You can try to verify this on your own to believe the equality.

Example 8.8

Several examples, because this is an important notion to grasp:

1. Let $p : \mathbb{A}^n \to \mathbb{A}^m$ be the projection map onto the first m coordinates (assume $n \geq m$). Then, the pullback map is the inclusion $k[X_1, \ldots, X_m] \to k[X_1, \ldots, X_n]$ where $X_i \mapsto X_i$.

2. Consider again the map $f : X \to Y$, where $X = \mathbb{A}^1$ and $Y = Z(y - x^2)$, given by $t \mapsto (t, t^2)$. The induced pullback is $f^* : k[X,Y]/(Y - X^2) \to k[T]$ where $\overline{X} \mapsto T$. This, you can check, is an isomorphism.

3. Consider now the map $t \mapsto (t^2, t^3)$ (this is from the non-isomorphism example above, Example 8.5). The corresponding pullback is

$$f^* : k[X,Y]/(Y^2 - X^3) \to k[T]$$
$$\overline{X} \mapsto T^2$$
$$\overline{Y} \mapsto T^3.$$

   This is **not** an isomorphism (just like how f itself was not an isomorphism), since T is not in the image of this map!

8.2 Correspondence Between Algebraic Sets and Algebras

We already knew that algebraic sets are in one-to-one correspondence between radical ideals, and by taking the quotient of the ideals in the polynomial ring, we've seen in class/psets that we have a correspondence between the algebraic sets (over k) and finitely-generated k-algebras. This correspondence goes deeper, though, now that we have a correspondence between morphisms of algebraic sets with morphisms of k-algebras.

Proposition 8.9

Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be algebraic sets. Then, there exists a one-to-one correspondence between morphisms $f : X \to Y$ and k-algebra homomorphisms $A(Y) \to A(X)$ given by $f \mapsto f^*$.

In other words, we have an equivalence between the category of algebraic sets over k and the category of finitely generated k-algebras. (For category-theory lovers: the map (**functor**) between the categories taking X to $\mathcal{O}_X(X) = A(X)$ is called the "global section" functor, since we're taking global sections, and it is a contravariant functor.) And in general, it is a lot easier to work with algebras than with some algebraic set, which doesn't have much structure to work with.

Proof. We want to construct an inverse of $f \mapsto f^*$. Let $\varphi : A(Y) \to A(X)$ be a k-algebra homomorphism. We have $A(Y) = k[Y_1, \dots, Y_m]/I(Y)$ and $A(X) = k[X_1, \dots, X_n]/I(X)$; denote $y_i = \overline{Y_i}$. Define maps on each of the coordinates, so construct $f_i : X \to \mathbb{A}^1$ where $f_i = \varphi(y_i)$. This gives a map $f : X \to \mathbb{A}^m$ where $f = (f_1, \dots, f_m)$.

We need to check that this f in fact maps into Y. Indeed, if $x \in X$ and $g \in I(Y)$, then $g(f(x)) = g(\varphi(y_1)(x), \dots, \varphi(y_m)(x)) = \varphi(g)(x)$ since φ is a homomorphism. But $g \in I(Y)$, so this is simply 0, meaning $f(x) \in Z(I(Y)) = Y$.

It remains to verify that this is indeed the desired inverse, i.e. $\varphi = f^*$. Suppose $h \in A(Y)$. Then, $f^*(h) = h \circ f = \varphi(h)$ (again since φ is a homomorphism), which means $\varphi = f^*$ as desired. □

Corollary 8.10

$f : X \to Y$ is an isomorphism iff $f^* : A(Y) \to A(X)$ is an isomorphism.

Remark 8.11. Note that this correspondence absolutely fails in the projective scene. Whereas in the affine case, taking the coordinate rings recovers all information about the algebraic set, in the projective case, doing so loses *all* information, since the only regular maps in projective space are just the constant functions. (Convince yourself that this is true!)

8.3 Redefining Morphisms

We return to our focus on our morphisms.

Lemma 8.12

If $f : X \to Y$ is a morphism, then f is continuous in the Zariski topology, i.e. if $Z \subseteq Y$ is an algebraic set, then $f^{-1}(Z) \subseteq X$ is an algebraic set.

Proof. Suppose $Z = Z(g_1, \dots, g_r)$ for $g_i \in A(Y)$. Then, using the fact $g_i \circ f = f^* \circ g_i$, we have

$$f^{-1}(Z) = \bigcap_{i=1}^{r} f^{-1}(Z(g_i)) = \bigcap_{i=1}^{r} Z(f^*(g_i)) = Z(f^* g_1, \dots, Z(f^* g_r),$$

which is algebraic, so we conclude. □

We've seen, for affine sets X, Y, a continuous map $f : X \to Y$ is a morphism, i.e. $\forall\, g \in A(Y) = \mathcal{O}_Y(Y)$, we have $f^*(g) \in A(X) = \mathcal{O}_{X,x}$.

Proposition 8.13

Let $f : X \to Y$ be a continuous map. Then, the following are equivalent:

1. f is a morphism,
2. $\forall\, U \subseteq Y$ open, $f^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(f^{-1}(U))$,
3. $\forall\, x \in X$ and $\varphi \in \mathcal{O}_{Y,f(x)}$, $f^*(\varphi) \in \mathcal{O}_{X,x}$.

This is a remarkable statement, perhaps a bit too remarkable for the purposes of this class. In scheme theory, morphisms are defined by either condition (2) or (3). (These are actually morphisms of **ringed spaces**, but you don't need to be boggled down by more terminology for now.) This proposition is just saying that this more high-powered definition of morphisms in fact coincides with our definition of morphisms. The useful thing about the scheme-theory definition is that it *does not rely on global data*. We strictly work in the local setting, and somehow we obtain information about its global behavior.

Proof. (2) implies (1) is clear: just take $U = Y$. (3) implies (2) is also clear from the observation $\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x}$. (1) implies (3) is also not too bad! If $\varphi \in \mathcal{O}_{Y,f(x)} \subseteq k(Y) = Q(A(Y))$, then we can express $\varphi = g/h$ where $g, h \in A(Y)$ such that $h(f(x)) = f^*(h)(x) \neq 0$. But then this means $f^*\varphi = f^*g/f^*h \in \mathcal{O}_{X,x}$. □

Another reason why the above proposition is useful is because we can adopt this definition of morphism for any kind of variety, e.g. open subsets of affine varieties (**quasi-affine**), projective varieties, open subsets of projective varieties (**quasi-projective**).

Definition 8.14 (Morphisms for quasi-affine)**.** A continuous map $f : X \to Y$ between open subsets of affine varieties is a morphism if either (2), (3) of Proposition 8.13 holds.

8.4 Properties of Morphisms

Lemma 8.15
Let $f : X \to Y$ be a morphism of affine varieties. then,

1. If $Z \subset f(X)$ and $f^{-1}(Z)$ is irreducible, then Z is irreducible.
2. $\overline{f(X)}$ is irreducible.

Remark 8.16. Note that the closure is necessary, since the image of a closed subset is not necessarily closed. For instance, the projection of the hyperbola $Z(xy-1)$ into the x-axis is $\mathbb{A}^1 - \{0\}$, which is open.

Remark 8.17. We will discuss this result more when we get to Chevalley's Theorem, which is a really important result in algebraic geometry. So keep this in mind!

Proof. We breeze through a proof of (1); the proof for (2) is of similar flavor. Suppose $Z = Z_1 \cup Z_2$. Then, $f^{-1}(Z) = f^{-1}(Z_1) \cup f^{-1}(Z_2)$, so WLOG $f^{-1}(Z) = f^{-1}(Z_1)$. This means $Z = Z_1$, done. □

Proposition 8.18
Let f be a morphism of affine varieties. If $f : X \to Y$ is surjective, then $f^* : A(Y) \to A(X)$ is injective.

Proof. Just follows by definition. If $g \in A(Y)$ and $f^*(g) = g \circ f = 0$, we must have $g = 0$ since f is surjective. □

However, the converse of this is false. Consider the pullback map $f^* : k[T] \to k[X, Y]/(XY - 1)$ mapping $T \mapsto X$. This is injective, but the projection of the hyperbola onto the x-axis is not surjective. (0 is missing.) We have a special name for when f^* is injective:

Definition 8.19 (Dominant). $f : X \to Y$ is **dominant** if $\overline{f(X)} = Y$.

Exercise 8.20. (Homework) f is dominant iff f^* is injective.

9 02/27 - Projective Varieties

We talked about last time (e.g. Corollary 8.10, Proposition 8.18) how we could nicely recover information about the morphism of varieties given the ring homomorphism between coordinate rings, and vice versa. We'll continue with this:

Proposition 9.1

Let $f : X \to Y$ be a morphism of affine varieties. Then, $f^* : A(Y) \to A(X)$ is surjective iff f is an isomorphism onto its image.

Proof. Suppose f is an isomorphism onto its image; let $Z = \operatorname{Im} f \subset Y$. This is given by the commutative diagram on the left, which induces the one on the right:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\sim}\downarrow & \nearrow & \\ Z = \operatorname{Im} f & & \end{array} \qquad\qquad \begin{array}{ccc} A(X) & \xleftarrow{f^*} & A(Y) \\ {\scriptstyle\sim}\uparrow & \swarrow & \\ A(Z) & & \end{array}$$

We have $Z \subseteq Y \implies I(Y) \subseteq I(Z)$, hence $A(Y) \twoheadrightarrow A(Z)$ and thus f^* is surjective.

For the forward direction, denote $Z = \overline{\operatorname{Im} f}$, so we can think of $f : X \to Z \hookrightarrow Y$. The map $X \to Z$ is dominant, so by a homework problem (Exercise 8.20), $A(Z) \to A(X)$ is injective, so $A(Z) \to A(X)$ is an isomorphism. By Corollary 8.10, $X \to Z$ is an isomorphism, and we conclude. □

9.1 Revisiting Projective Space

Algebraic geometry is all about the projective, so it's about time we talk about projective varieties.

Definition 9.2 (Projective Space)**.** The **projective space** $\mathbb{P}^n$ over field k is the set of lines throguh the origin in $\mathbb{A}_k^{n+1}$. Equivalently, it is the quotient $(\mathbb{A}^{n+1} \setminus \{0\})/k^\times$ via the action $\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$.

We notate the point in $\mathbb{P}^n$, which again represent equivalence classes of $\mathbb{A}^{n+1}$ modulo scaling, by $(x_0 : \cdots : x_n)$, and these are called the **homogeneous coordinates** on $\mathbb{P}^n$. Note that in homogeneous coordinates, the "value" x_i is not well-defined, but the quotient x_i/x_j **is** well-defined, since it is invariant under taking scalars. This is an important notion we take advantage of in more advanced algebraic geometry.

Remark 9.3. Over $\mathbb{C}$, $\mathbb{P}^n$ is compact in the classical topology! In fact, we have a surjective map $S^{2n+1} \twoheadrightarrow \mathbb{P}^n$. $(S^{2n+1} = \{(a_0, \dots, a_n) \in \mathbb{C}^{n+1} \mid |a_0|^2 + \cdots + |a_n|^2 = 1\}$. This is in fact a finite map, where the fibers are the two antipodal points.

When we were first talking about projective space (from Lecture 3), we stated the importance of homogeneous polynomials. We'll redefine:

Definition 9.4 (Homogeneous Polynomials)**.** A polynomial $f \in k[x_0, \dots, x_n]$ is **homogeneous** of degree d if $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$ for all $\lambda \in k$. Equivalently, f is a sum of monomials of degree d in the x_i's.

Definition 9.5 (Homogeneous Ideal)**.** An ideal $I \subseteq k[x_0, \ldots, x_n]$ is **homogeneous** if it can be generated by homogeneous polynomials (not necessarily of the same degree).

As a quick example, the ideal $(x, x + y^2)$ is not generated by homogeneous polynomials a priori, but we see that it is the same ideal as (x, y^2), so it is homogeneous. Similarly, $(x^2, x^3 + y^2)$ is a homogeneous ideal.

Equivalently, we can say an ideal I is homogeneous iff $\forall\, f \in I$, if we write $f = \sum_d f_d$ (f_d is the degree-d part of f), then $f_d \in I$ for all d.

9.2 Projective Varieties

We'd like to have ways to relate the affine picture with the projective setting. Let's work towards this:

Definition 9.6 (Porting over from affine stuff)**.** Note all ideals must be homogeneous.

1. Let $I \subseteq k[x_0, \ldots, x_n]$ be a homogeneous ideal. Then, we can define the **zero-set** of I:
$$Z(I) = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid f(x_0, \ldots, x_n) = 0\, \forall\, f \in I\}.$$
These are the **algebraic sets** of $\mathbb{P}^n$.

2. If $X \subset \mathbb{P}^n$ is any subset, we call
$$I(X) = \langle f \in k[x_0, \ldots, x_n] \text{ homogeneous} \mid f(x) = 0 \rangle$$
as the **ideal of X**.

3. The **homogeneous coordinate ring** of $X \subseteq \mathbb{P}^n$ is
$$S(X) := k[x_0, \ldots, x_n]/I(X).$$
This is a **graded ring** (here it is graded over $\mathbb{N} = \mathbb{Z}_{\geq 0}$). To not clutter up this definition, I define graded ring below. Letting R_d be the degree-d part of R, we have $S(X) = \bigoplus_d S(X)_d$.

Definition 9.7 (Graded ring)**.** A **($\mathbb{Z}_{\geq 0}$-)graded ring** is a ring R such that the underlying additive group is a direct sum of abelian groups R_i such that $\forall\, i, j \in \mathbb{Z}_{\geq 0}$, $R_i \cdot R_j \subset R_{i+j}$.

The most canonical example, and the one we'll probably only use in this course, is the ring of polynomials forms a graded ring, where the grading is just the degree of the homogeneous polynomials. (Check that if $S = R[x_0, \ldots, x_n]$, then $S = \bigoplus_{d \geq 0} S_d$.)

The following will look very familiar (see Lemma 5.2):

Proposition 9.8

We have the following:

1. If $I_1 \subseteq I_2 \subseteq k[x_0, \dots, x_n]$ are homogeneous ideals, then $Z(I_2) \subseteq Z(I_1)$.
2. If $\{I_i\}_{i \in I}$ is a family of homogeneous polynomials, then
$$\bigcap_{i \in I} Z(I_i) = Z\left(\bigcup_{i \in I} I_i\right).$$
3. If I_1, I_2 are homogeneous ideals, then $Z(I_1) \cup Z(I_2) = Z(I_1 \cdot I_2)$.

Consequently, just like in the affine case, we can define the **Zariski topology** on $\mathbb{P}^n$ where the closed sets are given by the projective algebraic sets in $\mathbb{P}^n$. Furthermore, for any $X \subseteq \mathbb{P}^n$, the Zariski topology on X is the induced topology from $\mathbb{P}^n$. And now for the golden word...

Definition 9.9 (Projective Variety)**.** A **projective variety** is an irreducible closed subset of $\mathbb{P}^n$.

The notion of dimension is the same as for $\mathbb{A}^n$.

Exercise 9.10. (Homework) $\dim S(X) = \dim X + 1$.

Example 9.11 (Projective Varieties)

Although we'll provide relatively simple examples, projective varieties are terrifying objects in the sense that we *really* don't understand them. For example, it took really difficult work from Griffiths and Harris to find a variety carved out by degree-3 polynomials in $\mathbb{P}^4$ which is not rational, and it is still unknown if such varieties in $\mathbb{P}^5$ is rational in general. But anyways,

1. If $F \in k[x_0, \dots, x_n]$ is homogeneous of degree d, then $Z(F) \subset \mathbb{P}^n$ is a **hypersurface of degree** d.
2. The image of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ given by $(x_0 : x_1) \mapsto (x_0^n : x_0^{n-1}x_1 : \cdots : x_1^n)$ is called the **rational normal curve of degree** n in $\mathbb{P}^n$. A really famous example is the **twisted cubic**, given by the map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ defined above. It is the intersection of three quadrics: $Y_0Y_3 = Y_1Y_2$, $Y_1^2 = Y_0Y_2$, and $Y_2^2 = Y_1Y_3$. These quadrics are the determinants of the 2×2 minors of
$$\begin{pmatrix} Y_0 & Y_1 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{pmatrix}.$$

   Try to generalize this for the degree n rational normal curve! And just to highlight the importance of this example, here's spewing nonsense: this is an example of a **determinantal variety** and the smallest example of a **Veronese embedding**.

9.3 Proving Affine Results for Projective

We want to construct the really nice correspondences we have in the affine case.

Definition 9.12 (Affine cone)**.** An affine algebraic set $Y \subseteq \mathbb{A}^{n+1}$ is called a **cone** if $\forall \lambda \in k$, $(x_0, \ldots, x_n) \in Y \iff (\lambda x_0, \ldots, \lambda x_n) \in Y$. If $X \subseteq \mathbb{P}^n$ is an algebraic set, we define the **affine cone over X** as

$$C(X) = \{(x_0, \ldots, x_n) \in \mathbb{A}^{n+1} \mid (x_0 : \cdots x_n) \in X\} \cup \{0\} \subseteq \mathbb{A}^{n+1}.$$

Note that 0, the vertex of the cone, is the intersection of a bunch of lines, so it is a singular point. This point contains a *lot* of data; it is a really subtle thing, and to be honest I don't know much about it, but broadly, geoemtric information about X is encoded in the commutative algebra that goes on around the vertex 0.

For a geometric picture of the affine cone, basically take the union of all the lines in $\mathbb{A}^{n+1}$ represented by the homogeneous coordinates $(x_0 : \cdots : x_n)$. I wish I could draw this out, but my LaTeX skills stop here. For another reformulation, let $I \subseteq k[x_0, \ldots, x_n]$ and $X = Z(I)$ in the projective sense. Then, $C(X) = Z(I)$ in the affine case, where we forget that I is a homogeneous ideal.

Now we reconstruct Nullstellensatz for the projective case.

Proposition 9.13 (Projective Nullstellensatz)
We have the following:

1. If $X_1 \subseteq X_2 \subseteq \mathbb{P}^n$ are algebraic sets, then $I(X_2) \subseteq I(X_1)$.
2. For any algebraic set $X \subseteq \mathbb{P}^n$, we have $Z(I(X)) = X$.
3. For any homogeneous ideal $J \subseteq k[X_0, \ldots, X_n]$ such that $Z(J) \neq \emptyset$, we have $I(Z(J)) = \sqrt{J}$.
4. For any homogeneous ideal $J \subseteq k[X_0, \ldots, X_n]$ such that $Z(J) = \emptyset$, we have either $J = (1)$ or $\sqrt{J} = (X_0, \ldots, X_n)$.

Equivalently, we can combine (3) and (4) by saying $Z(J) = \emptyset \iff (X_0, \ldots, X_n)^r \subseteq J$ for some $r \geq 0$.

So we see that the ideal $(X_0, \ldots, X_n)$ is stopping us from having a really nice time/getting really close to Nullstellensatz in the affine case. Consequently, we give it a very derogatory name: the **irrelevant ideal**, which is a bad name choice since it is actually very relevant. Spitefulness never solves problems... (The real reason for the name irrelevant is because the ideal gives no geometric information in $\mathbb{P}^n$.)

Proof. (1) and (2) follow from the affine case, as does the $\supseteq$ inclusion for (3). To prove (3), we will reduce to the affine case. If $Z(J) \neq \emptyset$, then $C(Z(J)) \neq \emptyset$ (they are given by the same ideal), so we can apply affine Nullstellensatz to get $I(Z(J)) = \sqrt{J}$.

For (4), if $Z(J) = \emptyset$, then in $\mathbb{A}^{n+1}$ (we forget J is homogeneous), we have either $Z(J) = \emptyset$ or $Z(J) = \{0\}$. Thus, either $J = (1)$ or $\sqrt{J} = (X_0, \dots, X_n)$, which is what we wanted. $\square$

10 03/01 - More on Projective Varieties

Happy March!... wait, we're already in March??

To make the correspondences implied by the Projective Nullstellensatz explicit, we have the same correspondences as in the affine case (algebraic sets with radical ideals, varieties with prime ideals, closed points with maximal ideals), except we add the condition that all ideals are homogeneous.

10.1 Veronese Embedding and Segre Map

Here are two really important examples of varieties determined by prime ideals:

The first is the **Veronese embedding**. Let $n, d > 0$, and $N = \binom{n+d}{n} - 1$. Consider the mapping

$$\begin{aligned} \nu_d : \mathbb{P}^n &\to \mathbb{P}^N \\ (x_0 : \dots : x_n) &\mapsto (x_0^d : x_0^{d-1}x_1 : \dots : x_n^d), \end{aligned}$$

where the monomials are ordered lexicographically. We denote the monomials as P_i, $1 \leq i \leq N$. Then, we have a map

$$\begin{aligned} \varphi : k[Y_0, \dots, Y_N] &\to k[X_0, \dots, X_n] \\ Y_i &\mapsto P_i. \end{aligned}$$

> **Exercise 10.1.** (Combinatorics) Verify that there are exactly $\binom{n+d}{n}$ monomials of degree d in $x_0, \dots, x_n$. (Many easier arguments, but one thing to note is that this corresponds with the dimension of $\operatorname{Sym}^d V$ where $\dim V = n + 1$.)

The relations of the monomials are determined by $\ker \varphi$. Thus, $\operatorname{Im} \nu_d$ is cut out by the polynomials in $\ker \varphi = I$, a prime ideal.

Example 10.2 (Twisted cubic as Veronese)

The famous twisted cubic (which we've mentioned many times in class by now) is the image of the map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ where $(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$.

You will prove many things about the Veronese embedding in your homework, including ν_d is a homeomorphism onto its image. These maps are nice because it turns all equations linear. For instance, if we want to study conics $Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 = 0$, then a priori the equation is quadratic in x, y, z. But if we consider the Veronese embedding $\nu_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ where $(x : y : z) \mapsto (x^2 : xy : xz : y^2 : yz : z^2)$, then suddenly the equation for the conic is linear in our monomials.

But perhaps you're still not super convinced that these actually matter. To convince you with something more concrete, the Veronese embeddings $\nu_n(\mathbb{P}^1) \hookrightarrow \mathbb{P}^n$ (the rational normal curve) and $\nu_2(\mathbb{P}^2) \hookrightarrow \mathbb{P}^5$ (Veronese surface) are examples of varieties of **minimal degree**. We will talk about this later.

The second important example we'll talk about is the **Segre map**, also featured in your homework. Let $n, m > 0$, and $N = nm + n + m = (n+1)(m+1) - 1$. Consider the map

$$\begin{gathered}\varphi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N \\ ((x_0 : \cdots : x_n), (y_0 : \cdots : y_m)) \mapsto (\cdots : x_i y_j : \cdots)_{ij}.\end{gathered}$$

Order the monomials lexicographically. Then, we can take the homomorphism

$$\begin{aligned} f : k[Z_{11}, \ldots, Z_{ij}, \ldots, Z_{nm}] &\to k[X_0, \ldots, X_n, Y_0, \ldots, Y_m] \\ Z_{ij} &\mapsto X_i Y_j. \end{aligned}$$

The kernel $\ker f$ realizes $\mathbb{P}^n \times \mathbb{P}^m$ as a projective variety in $\mathbb{P}^N$!

Example 10.3

The most famous example is also the simplest one. Take $n = m = 1$ and the map

$$\begin{gathered} Q : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \\ (x_0 : x_1), (y_0 : y_1) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1). \end{gathered}$$

The image is a hypersurface given by the equation $Y_0 Y_3 - Y_1 Y_2 = 0$. We call this the **quadric surface in $\mathbb{P}^3$**. To practice more with this, look at Exercise 2.15 in Hartshorne.

10.2 Functions and Morphisms on Projective Varieties

Definition 10.4 (Field of rational functions)**.** Let X be a projective variety and $S(X) = k[X_0, \ldots, X_n]/I(X) = \bigoplus_d S(X)_d$. ($S(X)_d$ is the degree-d part of the graded ring.) The **field of rational functions** is

$$K(X) = \{f/g \mid f, g \in S(X)_d \text{ for some } d, g(x) \neq 0\}.$$

Note that we require $f, g \in S(X)_d$ and not just $S(X)$ in order for these rational functions to be well-defined on homogeneous coordinates.

Given this, we can talk about local information: for all $x \in X$, define

$$\mathcal{O}_{X,x} = \{f/g \in K(X) \mid g(x) \neq 0\}$$

and $\forall U \subseteq X$ open,

$$\mathcal{O}_X(U) = \bigcap_{x \in X} \mathcal{O}_{X,x}.$$

Now we can talk about morphisms as in Proposition 8.13 between (open subsets of) projective varieties.

Remark 10.5. We call open subsets of projective varieties as **quasi-projective varieties**.

Definition 10.6. A continuous map $f : X \to Y$ is a morphism if one of the following equivalent definitions holds:

1. $\forall U \subseteq Y$ open, $f^*(\mathcal{O}_Y(U)) \subseteq \mathcal{O}_X(f^{-1}(U))$.
2. $\forall x \in X$ and $\varphi \in \mathcal{O}_{Y,f(x)}$, $f^*(\varphi) \in \mathcal{O}_{X,x}$.
3. There exists an open cover $\{U_i\}_i$ of Y such that (1) holds for all U_i. (Note this is strictly weaker than (1).)

So information about the morphism is embodied in the open subsets, i.e. morphisms are determined by their local behavior. Following (3), we want to take a convenient open cover for our projective varieties. Luckily, we have one at hand.

The **standard affine open cover** consists of basically the complements of the hyperplanes. More specifically, we have opens $U_i = \{(x_0 : \cdots : x_n) \mid x_i \neq 0\} \subseteq \mathbb{P}^n$ which is isomorphic to $\mathbb{A}^n = \{(y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n) \mid y_0, \ldots, \hat{y}_i, \ldots, y_n \in k\}$. ($\hat{y}_i$ just means it is excluded.) You can think of the map $U_i \xrightarrow{\sim} \mathbb{A}^n$ as $x_j \mapsto x_j/x_i = y_j$.

Example 10.7 (Affine Opens in Projective Varieties)

Let $X \subseteq \mathbb{P}^n$ be closed, and suppose $X = \bigcup_{i=0}^n X_i$ where $X_i = X \cap U_i$ are affine algebraic sets. We have $X = Z(F_1, \ldots, F_r)$ where the F_j's are homogeneous polynomials. Say $i = 0$. Then, we can let $f_j(x_1, \ldots, x_n) = F_j(1, x_1, \ldots, x_n)$ for $1 \leq j \leq r$, and denote $Y = Z(f_1, \ldots, f_r) \subseteq \mathbb{A}^n$. We now have a map from affine opens in X to the affine variety Y:

$$\begin{aligned} \varphi : X_0 = X \cap U_0 &\to Y \\ (x_0 : \cdots : x_n) &\mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right), \\ \varphi^{-1} : Y &\to X_0 \\ (x_1, \ldots, x_n) &\mapsto (1 : x_1 : \cdots : x_n). \end{aligned}$$

So this is good and all, since we know what morphisms are between affine varieties and we can just reduce our scenario in the projective case to the affine case via taking the standard affine open cover, yada yada. But this relies on taking local information, and in projective space, there exists better global statements.

Lemma 10.8
Let $X \subseteq \mathbb{P}^n$ be a projective variety, and $F_0, \ldots, F_m \in k[X_0, \ldots, X_n]$ be homogeneous polynomials of the *same* degree such that $\forall\, x \in X$, $\exists\, j$ such that $F_j(x) \neq 0$. Then, the mapping

$$\begin{aligned} f : X &\to \mathbb{P}^m \\ x &\mapsto (F_0(x) : \cdots : F_m(x)) \end{aligned}$$

is a morphism.

Admittedly, we prove this by taking an affine open cover, but the result is global, which is nice.

Proof. f is well-defined set theoretically because the F_i's are homogeneous of the same degree. It is also continuous, since it is given by polynomials. Consider the standard open cover $\{V_i\}_{0 \leq i \leq m}$ of $\mathbb{P}^m$, where $V_i = (y_i \neq 0)$. Consider $U_i = f^{-1}(V_i)$. Then, the restriction of f onto the U_i's look like

$$\begin{aligned} f|_{U_i} : U_i &\to V_i \\ x &\mapsto \left(\frac{F_j(x)}{F_i(x)}\right)_{j=0,\ldots,\hat{i},\ldots,n}. \end{aligned}$$

By definition, for $x \in U_i$, we have $F_i(x) \neq 0$, so $f|_{U_i}$ is regular. □

Definition 10.9 (Bihomogeneous)**.** A polynomial $F \in k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ is called **bihomogeneous** of **bi-degree** (d, e) if $\forall\, \lambda, \mu \in k$,

$$F(\lambda x_0, \ldots, \lambda x_n, \mu y_0, \ldots, \mu y_m) = \lambda^d \mu^e F(x_0, \ldots, x_n, y_0, \ldots, y_m).$$

In other words, F will look homogeneous of degree d if you fix all the y_j's, and homogeneous of degree e if you fix all the x_i's. A bihomogeneous polynomial, therefore, defines a subset of $\mathbb{P}^n \times \mathbb{P}^m$. Turns out that bihomogeneous polynomials define all closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$.

Proposition 10.10
A closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ is exactly the zero locus of a collection of bihomogeneous polynomials.

Remark 10.11. Note here this is the topology induced by the Segre map, **not** the product topology. The product topology is bad for our purposes! If we took the product topology, then a lot of curves which we expect to be closed will not be closed.

Proof. The closed subsets are of the form $\varphi_{n,m}^{-1}(Z)$ where $Z \subseteq \mathbb{P}^N$ is closed and $\varphi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ is the Segre map. We have $Z = Z(F_1, \dots, F_r)$ where the F_i's are homogeneous polynomials of degree d_i over the variables Z_{ij}. Taking the Z_{ij}'s under the map $k[\dots, Z_{ij}, \dots] \to k[X_0, \dots, X_n, Y_0, \dots, Y_m]$, we can define each F_k as defined over the X, Y variables to get $F_k(X_iY_j)$ a bihomogeneous polynomial of bidegree (d_k, d_k). We now invoke the following exercise:

> **Exercise 10.12.** Suppose G is bihomogeneous of bidegree (d, e). Then, $Z(G)$ is the same as the zero locus of $X_i^e Y_j^d G$ for all i, j.

Proof of this exercise is left to the reader (just think about it for a little bit). This result finishes the proof because □

Example 10.13

Revisit the Segre map $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ where $((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$. As noted before, the image is a hypersurface defined by $Y_0Y_3 - Y_1Y_2 = 0$ in $\mathbb{P}^3$. ($Y_0 = x_0y_0$, $Y_1 = x_0y_1$, etc.) We also have the twisted cubic given by the Veronese embedding $\nu_3 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. We see that $C = \nu_3(\mathbb{P}^1)$ is given by the equations $Y_0Y_3 - Y_1Y_2 = 0$, $Y_1^2 - Y_0Y_2 = 0$, and $Y_2^2 - Y_1Y_3 = 0$. Call the surfaces carved out by these equations as Q, Q', Q''. We note $Q = \varphi_{11}(\mathbb{P}^1 \times \mathbb{P}^1)$, and we have $C = Q \cap Q' \cap Q''$.

Looking at Q' and Q'' in terms of x_i, y_j, we have $(x_0y_1)^2 = (x_0y_0) \cdot (x_1y_0)$ and $(x_1y_0)^2 = (x_0y_1) \cdot (x_1y_1)$. Combining, this gives us $x_1y_0^2 = x_0y_1^2$, which is a bihomogeneous polynomial of bidegree $(1, 2)$. This gives us a unique curve on the quadric surface.

11 03/06 - Projective Varieties are Complete

11.1 Completeness of Projective Varieties

We have some nice topological properties for projective varieties. First, a projective variety is **separated**.

Lemma 11.1

If X is a projective variety, the diagonal $\Delta = \Delta_X := \{(x, x) \mid x \in X\} \subseteq X \times X$ is closed.

Proof. We have $\Delta_X = \Delta_{\mathbb{P}^n} \cap (X \times X)$. Thus, it suffices to show $\Delta_{\mathbb{P}^n}$ is closed. But $((x_0 : \cdots : x_n), (y_0 : \cdots : y_n)) \in \Delta_{\mathbb{P}^n}$ iff the rank of

$$\begin{pmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{pmatrix}$$

is at most 1. But this is equivalent to saying $x_i y_j = x_j y_i$ for all i, j, which are bihomogeneous of bidegree $(1, 1)$, so it forms a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$. The conclusion follows. □

We now introduce the notion of a complete variety, which is a really important property.

Definition 11.2 (Complete)**.** An algebraic variety Y is **complete** if for all algebraic varieties X, we have that $p_2 : Y \times X \to X$ is closed, i.e. images of closed sets are closed.

Theorem 11.3

Every projective variety is complete.

We will prove this after spring break, once we talk a bit about blow-ups. This will give us a trick to reduce Y to $\mathbb{P}^1$, which makes life a lot easier. But for now, we take for granted.

Example 11.4

A non-example. If $X = (xy = 1) \subseteq \mathbb{A}^2$ and $p_2 : \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is given by $(x, y) \mapsto x$, then we can note $p_2(X) = \mathbb{A}^1 \setminus \{0\}$ is not closed in $\mathbb{A}^1$, so $\mathbb{A}^1$ is not complete.

This theorem is *really significant*, as we'll now see. Many of these corollaries we've hoped to be true turn out to hold given the theorem!

Corollary 11.5

If X is a projective variety, then every morphism $\varphi : X \to \mathbb{P}^n$ is closed. (Consequently, any morphism $\varphi : X \to Y \subseteq \mathbb{P}^n$ is closed.)

Proof. The inclusion map $\Gamma_\varphi : X \hookrightarrow X \times \mathbb{P}^n$ mapping $x \mapsto (x, \varphi(x))$ is closed, and by our Theorem, $X \times \mathbb{P}^n \xrightarrow{p_2} \mathbb{P}^n$ is closed, so $\varphi : X \xhookrightarrow{\Gamma_\varphi} X \times \mathbb{P}^n \xrightarrow{p_2} \mathbb{P}^n$ is closed! □

Corollary 11.6

Every regular function on a projective variety X is constant.

This is the algebraic geometry analogue to the complex analysis fact that any holomorphic function on all of the complex plane is constant.

Proof. From above (Corollary 11.5), the image of $f : X \to \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ is closed. Since X is a projective variety, it is irreducible, so $\overline{f(X)} = f(X)$ is also irreducible (Lemma 8.15). But then $f(X)$ must be a point, since it is irreducible in $\mathbb{P}^1$. □

The corollaries just keep coming.

Corollary 11.7

The only projective subvarieties of affine varieties are the points.

Proof. Let X be affine, and $A(X)$ the algebra of functions on X. We have $A(X) = k$ only if $I(X)$ is maximal, i.e. X is a point, and we conclude. $\square$

Corollary 11.8

Let $X \subseteq \mathbb{P}^n$ be a projective variety different from a point. Then, if $H \subseteq \mathbb{P}^n$ is any hypersurface, then $X \cap H \neq \emptyset$.

Proof. Assume there exists $H = Z(F) \subseteq \mathbb{P}^n$, where F is homogeneous of degree d, such that $H \cap X = \emptyset$. Note then for all G homogeneous of degree d, G/F is a regular function on X, so G/F is constant on X by Corollary 11.6. But then under the Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, X will just map to a point, which is a contradiction since the Veronese embedding is an inclusion. $\square$

11.2 Applications to Varieties with Group Structure

Theorem 11.9 (Rigidity Lemma, I)

Let $\varphi : V \times W \to Z$ be a regular map of (quasi-projective) varieties such that $\exists\, v_0 \in V, w_0 \in W, z_0 \in Z$ such that $\varphi(\{v_0\} \times W) = \varphi(V \times \{w_0\}) = \{z_0\}$ and V is complete. Then, $\varphi(V \times W) = \{z_0\}$.

Proof. Let's start with the information about completeness that we have. $p_2 : V \times W \to W$ is closed by completeness of V. Let $z_0 \in U \subseteq Z$ be an affine open neighborhood, and define

$$T := p_2(\varphi^{-1}(Z \setminus U)).$$

Since $Z \setminus U$ is closed, the preimage is closed, so T is closed since p_2 is a closed map. We have $W \setminus T = \{w \in W \mid \varphi(V \times W) \subseteq U\}$. Since $w_0 \in W \setminus T$, we know $W \setminus T \neq \emptyset$, so we can choose some $w \in W \setminus T$.

Noting that $V \times \{w\}$ is a complete projective variety and $\varphi(V \times \{w\}) \subseteq U$ for affine open U, Corollary 11.7 tells us that $\varphi(V \times \{w\})$ is a point for all $w \in W \setminus T$. Even better, noting that $\varphi(\{v_0\} \times \{w\}) = \{z_0\}$ and $v_0 \in V$, we have $\varphi(V \times \{w\}) = \{z_0\}$ for all $w \in W \setminus T$, so it is constant on $V \times (W \setminus T) \subseteq V \times W$. But this is an open dense subset, so φ is constant on all of $V \times W$, as desired. $\square$

As mentioned by the section title, this has some nice applications to algebraic groups.

Definition 11.10 (Algebraic Group)**.** An **algebraic group** is a quasi-projective variety G with a group structure such that $G \times G \to G$ mapping $(g, g') \mapsto gg'$) and $G \to G$ ($g \mapsto g^{-1}$) are both morphisms of varieties.

Example 11.11

The additive and multiplicative groups $\mathbb{G}_a := (k, +)$ and $\mathbb{G}_m := (k^\times, \cdot)$ are varieties with a group law. We also note that $\mathrm{GL}_n(k) \subset M_n(k) \cong \mathbb{A}^{n^2}$ is the complement of $(\det = 0)$, hence it is a variety. It has a group law under multiplication. Similarly, $\mathrm{SL}_n(k) \subset M_n(k)$ is the variety defined by $(\det = 1)$, so it too is an algebraic group.

Definition 11.12 (Abelian Variety)**.** An **abelian variety** is a projective algebraic group.

Example 11.13

Elliptic curves are abelian varieties! They are projective varieties, and they have a (really cool) group law.

The group operation for elliptic curves being commutative is clear once you define the operation, but it is not clear at all a priori that any projective algebraic group turns out to be abelian. We will prove this shortly, after we lay out the following result.

Corollary 11.14

Every morphism $\alpha : A \to B$ between abelian varieties is the composition of a group homomorphism and a translation.

A translation is a map $G \xrightarrow{t_a} G$ for some $a \in G$ such that $g \mapsto a + g$.

Proof. After a translation, we may assume $\alpha(0) = 0$. Define $\varphi : A \times A \to B$ given by $\varphi(a, a') = \alpha(a+a') - \alpha(a) - \alpha(a')$. Computing explicitly, we have $\varphi(\{0\} \times A) = \varphi(A \times \{0\}) = \{0\}$, so by the Rigidity Lemma (Theorem 11.9), $\varphi = 0$, i.e. α is a group homomorphism, as desired. □

Now for what we wanted:

Corollary 11.15

The group law on an abelian variety is commutative.

Proof. Slick proof. Consider the inverse map $\iota : A \to A$ where $a \mapsto a^{-1}$ and $0 \mapsto 0$. This is a morphism by definition of algebraic group. At the same time, by the above corollary (11.14), we have $(ab)^{-1} = a^{-1}b^{-1}$. But for groups in general, $(ab)^{-1} = b^{-1}a^{-1}$, so $ab = ba$, as desired. □

Note that the one fact that drove us up to here is Theorem 11.3. Everything just kinda fell like dominoes from it! Hopefully this is enough to convince you that it is a very important fact, and one that we'll prove after spring break.

11.3 Chevalley's Theorem

We're now going to shift gears a little bit. Let's look at an innocuous-looking morphism. Consider $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ where $(x, y) \mapsto (x, xy)$. (I would draw cool diagrams to accompany this, but alas.) The horizontal lines map to lines through 0 except for the y-axis, and the y-axis maps to the origin, and all other vertical lines maps to themselves. So the resulting picture is kinda like a bunch of vertical lines and lines through the origin with the y-axis missing and replaced by a huge origin. This is a really weird image.

> **Definition 11.16** (Locally Closed)**.** A **locally closed** subset of a topological space is the intersection of a closed set and an open set.

By "really weird image" I mean that it is not only neither open nor closed, but it is not locally closed.

> **Definition 11.17** (Constructible)**.** A finite union of locally closed subsets is a **constructible set**.

Luckily, though, $\operatorname{Im}\varphi$ is constructible, since it is the union of $\{(0, 0)\}$ and $\{\mathbb{A}^2 \setminus (x = 0)\}$. This is a very fundamental construction (no pun intended), and the one we truly care about:

Theorem 11.18 (Chevalley)
If $f : X \to Y$ is a morphism of quasi-projective varieties, then $f(X)$ is a constructible set.

(Usually, the theorem comes with more information about the dimensions of the fibers and such, but we will discuss that later.)

12 03/08 - Rational Maps

Today, we will introduce rational maps in the projective setting.

Example 12.1
Let V be a vector space over k, say $V = U \oplus W$. We have a projection map $p_W : V \to W$. Let $\mathbb{P}(V)$ be the set of all 1-dimensional subspaces of V. We see that since $V \cong k^{n+1}$ for some n, we have an isomorphism $\mathbb{P}(V) = \mathbb{P}^n$. Note $\mathbb{P}(U), \mathbb{P}(W) \subseteq \mathbb{P}(V)$. Note that

then, the "projection outside of $\mathbb{P}(U)$" given by $p : \mathbb{P}(V) \setminus \mathbb{P}(U) \to \mathbb{P}(W)$ is induced by p_W. This p is a morphism; verification is left as an exercise.

Example 12.2

As an example of the above, suppose $\dim U = 1$. Then, $\mathbb{P}(U)$ is just a point, say u, and we have $p : \mathbb{P}^n \setminus \{u\} \to \mathbb{P}^{n-1}$ as given by $p(x) = \overline{ux} \cap \mathbb{P}^{k-1}$. (Here, $\overline{ux}$ is the line through u and x.) One can give a smiliar description for arbitrary U; this is also left as an exercise.

Now we go ahead and define rational maps. Let X, Y be (quasi-projective) varieties. Consider the pairs (u, U) where $U \subseteq X$ is open and $u : U \to Y$ is a morphism. We define an equivalence relation $(u, U) \sim (v, V)$ if u and v coincide on $U \cap V$.

Definition 12.3 (Rational Map)**.** A **rational map** $X \to Y$ is an equivalence class of such pairs. We denote as $u : X \dashrightarrow Y$. We say that u is **defined** at $x \in X$ if it has a representative (u, U) such that $x \in U$. The **domain of definition** of u is the set of points where u is defined; this is an open subset of X.

Definition 12.4 (Rational Function)**.** A **rational function** is a rational map $X \dashrightarrow \mathbb{A}^1 = k$.

We should check that this definition of rational function matches our old one, an element in the field of fractions of the coordinate ring.

Proposition 12.5

Let X be a quasi-projective variety. Then, the rational functions on X form a field $K(X)$ which is a k-extension. If $\emptyset \neq U \subseteq X$ is open, then $K(U) = K(X)$. If X is affine, then $K(X) = Q(A(X))$. (This is the old definition.)

Proof. Note that any $v : V \to k$ is equivalent to $v|_{V \cap U} : V \cap U \to k$ given $V \cap U \neq \emptyset$ for $U \subseteq X$ open. This proves the first part. For the second, let $f/g \in Q(A(X))$. We associate this to a rational function $f/g : X \setminus Z(g) \to k$, where the domain here is open. This proves one inclusion $Q(A(X)) \subset K(X)$.

Now we start with $u : U \to k$ a regular function. Define $Y = X \setminus U \subseteq X$ a closed affine subset. Then, there exists a function h on X such that $h = 0$ on Y (e.g. consider any $h \in I(Y)$). Let $X_h = X \setminus Z(h)$ be the distinguished affine. We can associate $A(X_h)$ with $A(X)_h$, the latter being localization at the element h. (This is an exercise, which you probably did in some fashion on your homework.)

This means that on V, we may write $u = \frac{g}{h^p}$ for some $p \geq 0$. But now we have $g/h^p \in Q(A(X))$, so we have associated u with some element in $Q(A(X))$. We can check that these mappings are inverse to each other (left for you to check), and so we conclude. $\square$

12.1 Dominant Rational Maps

Recall that a morphism $\varphi : X \to Y$ is **dominant** if $\overline{\varphi(X)} = Y$. Although rational maps are not morphisms themselves, we can define dominance for them as well:

Definition 12.6 (Dominant rational map)**.** A rational map $u : X \dashrightarrow Y$ is **dominant** if it has a regular representative which is dominant.

Remark 12.7. If $u : X \dashrightarrow Y$ is *dominant*, we can compose it with another rational map $v : Y \dashrightarrow T$. In particular, for $T = k$, we can compose with a rational function $\varphi : Y \dashrightarrow k$. This induces a field homomorphism $u^* : K(Y) \hookrightarrow K(X)$, and this map is a k-extension. So dominance is a pretty important property to us, as we'll continue to see below.

Proposition 12.8

Let X, Y be (quasi-projective) varieties.

1. The correspondence given by $u \mapsto u^*$ gives a one-to-one mapping between the sets $\{\text{dominant rational maps } u : X \dashrightarrow Y\}$ and $\{k - \text{extensions } K(Y) \hookrightarrow K(X)\}$. (This is in fact an equivalence of categories, i.e. it is functorial, so the correspondence is nicer than at first glance.)

2. u^* is an isomorphism iff u induces an isomorphism between non-empty open sets $U \subseteq X$ and $V \subseteq Y$. We call u a **birational map**; in complex analysis, these would be bi-meromorphic functions. X, Y are birational if there exists a birational map between them.

Proof. Let $i : K(Y) \hookrightarrow K(X)$ be a field extension over k. We want to construct a rational map $u : X \dashrightarrow Y$. We may assume $X \subseteq \mathbb{A}^m, Y \subseteq \mathbb{A}^n$ are affine, so $K(X) = Q(A(X))$ and $K(Y) = Q(A(Y))$. Let $\{y_j\}$ be the generators of $A(Y)$ as a k-algebra, and let $i(y_j) = \frac{a_j}{b_j} \in K(X)$ where $a_j, b_j \in A(X)$. This induces a map of k-algebras

$$\begin{aligned} i : A(Y) &\hookrightarrow A(X)_{b_1 \cdots b_n} \\ y_j &\mapsto \frac{a_j}{b_j}. \end{aligned}$$

But $X_{b_1 \cdots b_n}$ is a distinguished open affine in X, so this gives us a morphism $u : X \setminus Z(b_1 \cdot \cdots \cdot b_n) \to Y$. Even better, as i is injective, one of your homework exercises tells us that u is dominant. $\square$

Definition 12.9 (Birational Morphism)**.** If a morphism $u : X \to Y$ is a birational map, we call it a **birational morphism**.

12.2 Aside: Rational Varieties

We can also slap on these adjectives to the varieties themselves:

> **Definition 12.10** (Rational, Unirational Variety)**.** A variety X is **rational** if it is birational to $\mathbb{P}^n$. A variety X is **unirational** if there exists a dominant rational map $u : \mathbb{P}^n \dashrightarrow X$.

Note that given our Proposition (12.8), these strictly properties in field theory: for rational, we just need $K(X)$ to be a purely transcendental extension of k, and for unirational, $K(X)$ is a subextension of k inside a purely transcendental extension.

But is defining unirational even necessary? Or are all unirational varieties just rational after all?

Is every unirational variety actually rational?

You may recall from the beginning of this class when we were talking about curves (which have transcendence degree 1), Lüroth's Theorem tells us that any subextension of a purely transcendental extension of degree 1 is the whole extension itself, so the answer for curves is a resounding yes. There is a remarkable result (very geometric proof) by Castelnuovo which tells us that the answer is yes for surfaces in general. This is one of the last results in Hartshorne, for those interested.

But our luck stops there. This already fails in dimension 3, the first counterexamples being a cubic 3-fold $X^3 \subset \mathbb{P}^4$ given by Clemens-Griffiths and a quartic 3-fold given by Iskovskih-Manin.

The answer is unknown for 4-folds $X^3 \subseteq \mathbb{P}^5$. This is a very active field of algebraic geometry, and Popa mentioned a bit more of the story in class, but whew it is a lot haha.

12.3 Blow-Ups

Let's start with a motivating example. Let's blow up some points in $\mathbb{P}^n$, mwahahaha.

Let $x_0 \in \mathbb{P}^n$, $H \subseteq \mathbb{P}^n$ hyperplane ($H \cong \mathbb{P}^{n-1}$) such that $x_0 \notin H$. Choose coordinates such that $x_0 = (0 : \cdots : 0 : 1)$ and $H = Z(x_n)$. Then, we have a rational map $p : (x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_{n-1})$.

When we blow-up a point, we're going to consider the point as the union of lines passing through that point. We do this by the following.

Consider the graph of p, given by

$$\mathbb{P}^n \times \mathbb{P}^{n-1} = \mathbb{P}^n \times H \supseteq \Gamma_p = \{(\underline{x}, \underline{y}) \mid \underline{x} \neq x_0, (x_0 : \cdots : x_{n-1}) = (y_0 : \cdots : y_{n-1})\}.$$

Take $\overline{\Gamma_p} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$. The equations defining $\overline{\Gamma_p}$ are given by $X_i Y_j = X_j Y_i$ for $0 \leq i, j \leq n$. We denote $\widetilde{\mathbb{P}^n} := \overline{\Gamma_p}$ as the **blow-up** of $\mathbb{P}^n$ at x_0.

We have projection maps $\pi : \widetilde{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ and $q : \widetilde{\mathbb{P}^n} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$. The first projection π is the blow-up map. The fibers of π look like:

- if $x \neq x_0$, then $\pi^{-1}(x) = \{x\}$
- if $x = x_0$, then $\pi^{-1}(x) = H = \mathbb{P}^{n-1}$.

This is what we are trying to get at geometrically: fill this in!

13 03/20 - Blow-Ups

This lecture and the next were recorded over Zoom since Prof Popa was not on campus. We will first discuss blow-ups of points in $\mathbb{P}^n$, which is a natural place to start because we have projection maps readily available to us.

13.1 Blow-Up of Projective Space

We talked about this setup at the end of last class, but we will repeat it here. First, choose some point $x_0 \in \mathbb{P}^n$, WLOG $x_0 = (0 : 0 : \cdots : 0 : 1)$, and choose some hyperplane $H \not\ni x_0$, WLOG $H = Z(x_n)$. Then, we have a projection map

$$\begin{aligned} p : \mathbb{P}^n \setminus \{x_0\} &\to H \simeq \mathbb{P}^{n-1} \\ (x_0 : \cdots : x_n) &\mapsto (x_0 : \cdots : x_{n-1}). \end{aligned}$$

Then, we can define the blow-up

$$\mathrm{Bl}_{x_0}(\mathbb{P}^n) = \widetilde{\mathbb{P}^n} := \overline{\Gamma_p} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$$

carved out by the equations $X_iY_j = X_jY_i$ for $0 \leq i, j \leq n-1$. (This is just saying that the first n coordinates must match between the $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ components. The blow-up $\widetilde{\mathbb{P}^n}$ has two natural projections: $\widetilde{\mathbb{P}^n} \xrightarrow{\pi} \mathbb{P}^n$, which is just the **blow-up map**, and $\widetilde{\mathbb{P}^n} \to \mathbb{P}^{n-1}$.

We can study the fibers of the blow-up map π. Over $x \neq x_0$, one can check that $\pi^{-1}(x) = x$ (given the $\mathbb{P}^n$-component, the homogeneous coordinates in $\mathbb{P}^{n-1}$ are determined). But over $x = x_0$, since $x_i = 0$ for $0 \leq i \leq n-1$, $\overline{\Gamma_p}$ is given by no nonzero equations, so $\pi^{-1}(x_0) = H = \mathbb{P}^{n-1}$.

The geometric intuition is along the lines of the following: if given x_0 and some $x \neq x_0$, then there is a unique line passing through the two points, whereas given just x_0, we can go in any direction. So this is blowing up x_0, as it extracts the point and replaces it with a copy of $\mathbb{P}^{n-1}$. In short, we are treating the points in $\mathbb{P}^{n-1}$ as lines in $\mathbb{P}^n$ passing through x_0.

Building on this correspondence between points in $\mathbb{P}^{n-1}$ and lines in $\mathbb{P}^n$ through x_0, we can think of $\widetilde{\mathbb{P}^n}$ as the set $\{(x, \ell) \mid x \in \ell\} \subseteq \mathbb{P}^n \times \mathbb{P}^{n-1}$. This is an example of an **incidence correspondence**.

Explicitly, we have that π thus induces a birational morphism $\pi : \widetilde{\mathbb{P}^n} \setminus H \to \mathbb{P}^n \setminus \{x_0\}$.

Looking at the other projection $q : \widetilde{\mathbb{P}^n} \to \mathbb{P}^{n-1}$, we observe that once we specify some line $[\ell] \in \mathbb{P}^{n-1}$, then its pre-image $q^{-1}([\ell])$ is just the line ℓ, hence is isomorphic to $\mathbb{P}^1$. We call this the "**tautological bundle**" because we're not really gaining anything new from this observation.

Exercise 13.1. (If you want to play around with this a little more) Let $\{U_i\}_{i=0}^{n-1}$ be the standard open affine cover of $\mathbb{P}^{n-1} = H$. Then,

$$q^{-1}(U_i) \cong U_i \times \mathbb{P}^1.$$

13.2 Blow-Up of Projective Variety

So that's the blow-up of $\mathbb{P}^n$, but in fact we can blow-up any subvariety of $\mathbb{P}^n$.

Definition 13.2 (Blow-up of Projective Variety)**.** Let $X \subseteq \mathbb{P}^n$ be a subvariety, $x_0 \in X$. The **blow-up of X at x_0** is

$$\widetilde{X} := \overline{\pi^{-1}(X \setminus \{x_0\})} \subseteq \widetilde{\mathbb{P}^n}.$$

Popa drew a nice diagram, which I have a hard time replicating here but I will do my best explaining in words.

Example 13.3

(describing Popa's picture in words) We can take a variety X in $\mathbb{P}^2$ like a cubic with a singular point x_0 (node). Then, the pre-image of $X \setminus \{x_0\}$ in $\widetilde{\mathbb{P}^n}$ is a 1-1 lifting, so it looks like a non-intersecting curve with two distinct points missing, where the two points lie over x_0. Taking the closure, you complete the curve. The two points lying over x_0 correspond to the two tangent lines of the curve at x_0 in $\mathbb{P}^2$.

This blow-up construction turns out to be intrinsic and not dependent on any embedding.

Fact 13.4. The birational morphism $\pi : \widetilde{X} \to X$ is independent of the choice of embedding $X \subseteq \mathbb{P}^n$.

13.3 Local Version

Looking locally in our projective space, we can now consider blow-ups of $\mathbb{A}^n$, and then for any subvariety of $\mathbb{A}^n$. Denote $O := (0, \ldots, 0) \in \mathbb{A}^n$. Then, we can define $\pi : \mathrm{Bl}_0(\mathbb{A}^n) = \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ as basically dehomogenizing the previous construction with respect to x_n. In particular, by thinking of $\mathbb{A}^n$ as a variety of $\mathbb{P}^n$, we can consider the composition map $\pi : \widetilde{\mathbb{A}^n} \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1} \twoheadrightarrow \mathbb{A}^n$, where $\widetilde{\mathbb{A}^n}$ is defined in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ by the equations $X_iY_j = X_jY_i$ for $0 \leq i, j \leq n-1$.

The arguments from above apply here. If $x \neq 0$ in $\mathbb{A}^n$, then $\pi^{-1}(x)$ is a single point, so $\pi : \widetilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism (one can think of it as just the identity), with inverse $\psi : (x_0, \ldots, x_{n-1}) \mapsto ((x_0, \ldots, x_{n-1}), (x_0 : \cdots : x_{n-1}))$. For the pre-image of 0, just as before, this would make all equations defining $\widetilde{\mathbb{A}^n}$ vanish, so $\pi^{-1}(0)$ is all of $\mathbb{P}^{n-1}$, i.e. the set of all $(0, (y_0 : \cdots : y_{n-1}))$. So we may interpret $\mathbb{P}^{n-1}$ here as the set of lines in $\mathbb{A}^n$ passing through the origin.

Cue Popa drawing yet another diagram which I cannot replicate. Luckily, the Zoom recording will be kept until the end of class, so you can review it there! But the idea is similar to the example I tried to illustrate, literally.

Actually, here is a picture I snagged from online. The bottom is the affine plane, which lifts to the blow-up. The horizontal lines represent the pre-images of the lines in $\mathbb{A}^2$ passing through the origin, and the vertical line is the fiber of 0.

Figure 1: Credit: Andreas Gathmann

Remark 13.5. This is just the picture over $\mathbb{R}$. The picture for the blow-up over $\mathbb{C}$ is a bit more complicated, but the idea is similar.

Just to give it a name, the codimension-1 subset $\pi^{-1}(0)$ is called the **exceptional divisor** (or **exceptional locus**) of π. ("Divisor" is a special term in algebraic geometry that we won't discuss in this class.)

Remark 13.6. Let L be a line. Then, $\overline{\pi^{-1}(L \setminus \{0\})}$ intersects $\pi^{-1}(0)$ in the point corresponding to L. As a consequence, $\widetilde{\mathbb{A}^n} \setminus \pi^{-1}(0) \cong \mathbb{A}^n \setminus \{0\}$ is dense in $\widetilde{\mathbb{A}^n}$, which makes it irreducible. The same follows for $\widetilde{\mathbb{P}^n}$.

We note, though, that $\widetilde{\mathbb{A}^n}$ is a bit cumbersome to deal with because it is neither affine nor projective. But for any variety, we can cover it with affine charts, and we know how to make computations on affines. Thus, to make any sort of computation, we will want to use affine coordinate charts. There is a standard way to cover the exceptional divisor $\mathbb{P}^{n-1}$ by the standard open affines $\cong \mathbb{A}^{n-1}$. We demonstrate on $\widetilde{\mathbb{A}^2} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$.

On $\mathbb{P}^1$, we have two affine opens: $U_0 = (Y_0 \neq 0)$ and $U_1 = (Y_1 \neq 0)$. If $Y_0 \neq 0$ (i.e. on U_0), then since we can write $(y_0 : y_1) = (1 : y_1/y_0)$, we are just considering coordinates in $\mathbb{A}^2 \times \mathbb{A}^1$ (the $\mathbb{A}^1$ is given by $u = y_1/y_0$). On here, $\widetilde{\mathbb{A}^2}$ is defined by $x_0 y_1 = x_1 y_0 \implies x_1 = x_0 \cdot u$. So on $U_0 \cap \widetilde{\mathbb{A}^2}$, the blow-up map is the map $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, u) \mapsto (x, xu)$. (We just represent (x_0, x_1) by just x since given x_0, we can uniquely determine x_1.)

Remark 13.7. This is exactly the same picture as the motivating example for Chevalley's Theorem! (This is given in the beginning of §11.3.) In particular, the image is not locally closed.

For $Y_1 \neq 0$ (i.e. on U_1), the coordinates on $\mathbb{A}^2 \times \mathbb{A}^1$ are given by x_0, x_1, and $v = y_0/y_1$. Thus, $\widetilde{\mathbb{A}^2}$ is given by $x_0 = v \cdot x_1$, and the blow-up map π on $U_1 \cap \widetilde{A^2}$ is roughly the same, taking $(v, y) \mapsto (vy, y)$.

Remark 13.8. Here, it's just a coincidence that the charts on $\widetilde{\mathbb{A}^2}$ are isomorphic to $\mathbb{A}^2$, but the procedure for determining the blow-up map on affine charts is the same in general.

Definition 13.9 (Transforms)**.** Let $Y \subseteq \mathbb{A}^n$ be a closed subset, and $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ is the blow-up map. The **total transform** of Y is $\pi^{-1}(Y)$, and the **proper transform** of Y is $\widetilde{Y} := \overline{\pi^{-1}(Y \setminus \{0\})}$.

The proper transform $\widetilde{Y}$ is basically our blow-up, as it is in Example 13.3.

Definition 13.10 (Blow-Up of Y)**.** If Y is a subvariety of $\mathbb{A}^n$ such that $0 \in Y$, then the blow-up of Y at 0 is $\pi : \widetilde{Y} \to Y$.

We will finish out the lecture with a couple of concrete calculations. We'll take curves in $\mathbb{A}^2$ and consider its blow-up on the standard affine charts to make computations upstairs. We begin with smooth curves.

Example 13.11 (Blow-up of smooth)

Let $Y = (y = x^2) \subset \mathbb{A}^2$. On $U_0 = (Y_0 \neq 0) \subseteq \mathbb{A}^2 \times \mathbb{A}^1$, coordinates are given by $y = ux$ (here, $\mathbb{A}^2$ has coordinates (x, y) and $\mathbb{A}^1$ has coordinate u). Then, $\pi^*(y - x^2) = ux - x^2 = x(u - x)$, so the pullback is the exceptional divisor $(x = 0)$ plus another curve/line transverse to $(x = 0)$.

On $U_1 = (Y_1 \neq 0)$, coordinates are given by $x = vy$, so $\pi^*(y - x^2) = y - v^2y^2 = y(1 - v^2y)$, so the pullback is the exceptional divisor $(y = 0)$ plus the curve given by $1 - v^2y$, for which $(y = 0)$ is asymptotic but never intersects. Extending to $\mathbb{P}^1$, though, the curve intersects the exceptional divisor at the point of infinity, given by the intersection point in the first affine chart.

Putting these two pictures together, the image of the blow-up map $\pi : \widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ consists of parabolas going through the origin. Considering the pre-image upstairs, $\pi^{-1}(0)$ is the exceptional divisor $\cong \mathbb{P}^1$, and for any parabola, its pre-image is a curve which intersects the exceptional divisor exactly in the point that corresponds to the tangent direction of the parabola.

Now, we advance things by introducing singular points, starting first with a node.

Example 13.12 (Blow-up with node)

Let $Y = (y^2 = x^2 + x^3) \subseteq \mathbb{A}^2$. On $U_0 = (Y_0 \neq 0)$, the coordinates are given by $y = ux$, so $\pi^*(y^2 - x^2 - x^3) = u^2x^2 - x^2 - x^3 = x^2(u^2 - 1 - x)$. Thus, the pullback consists of the exceptional divisor given by $x^2 = 0$, the doubled line, plus another curve intersecting it at $u = \pm 1$.

For $U_1 = (Y_1 \neq 0)$, the coordinates are given by $x = vy$, so $\pi^*(y^2 - x^2 - x^3) = y^2 - v^2y^2 - v^3y^3 = y^2(1 - v^2 - v^3y)$. The pullback consists of the exceptional divisor $(y^2 = 0)$, again a doubled line, plus a curve intersecting it at $v = \pm 1$.

Globally, we note that on $U_0 \cap U_1$, we have $u = \frac{1}{v}$, so the intersection points (± 1) must be the same. Thus, given $Y \subseteq \mathbb{A}^2$ in the image of π, the blow-up is given by the exceptional divisor $\pi^{-1}(0) = E \simeq \mathbb{P}^1$ and a curve which intersects E twice, at the points corresponding to the tangent directions of Y at the origin.

Remark 13.13. Although all of our examples have taken singularities and turned them into good ol' transverse intersections, this is not always the case with blow-ups. But it is a good start to dealing with singularities!

14 03/22 - Proof of Theorem 11.3

Today, we will use blow-ups to prove the incredible Theorem 11.3, a result which we used to prove a whole series of other nice results. To do this, we will do a brief overview of resultants and elimination theory.

14.1 Resultants and Elimination Theory

We will start simple by taking polynomials in just one variable. Let $f, g \in k[Y]$, $\deg f = d$, and $\deg g = e$. We want to find a way to determine if f and g have a common nontrivial factor.

Resultants give us a nice tool for this. Suppose f and g do have some common factor, so $f = hf'$ and $g = hf'$. Seeing that $f, g \mid hf'g'$, we have that there must exist a polynomial of degree $\leq d + e - 1$ divisible by both f and g. Denote

$$V_f = \{\text{polynomials of deg} \leq d + e - 1 \text{ divisible by } f\}$$
$$V_g = \{\text{polynomials of deg} \leq d + e - 1 \text{ divisible by } g\}.$$

V_f and V_g are both vector spaces over k, and from our above observation, they intersect nontrivially. We can find a spanning list for each vector space: V_f is spanned by $f, Y \cdot f, Y^2 \cdot f, \ldots, Y^{e-1} \cdot f$ and V_g is spanned by $g, Y \cdot g, Y^2 \cdot g, \ldots, Y^{d-1} \cdot g$. Nontrivial intersection between these two vector spaces means that the above spanning vectors are linearly dependent.

We have a nice way of formulating this in linear algebra terms. Let $f = a_0 + a_1Y + \cdots + a_dY^d$ and $g = b_0 + b_1Y + \cdots + b_eY^e$. Linear dependence of the spanning vectors is equivalent to saying that the following determinant is equal to 0:

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_d & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{d-1} & a_d & 0 & \dots & 0 \\ 0 & 0 & a_0 & \dots & a_{d-2} & a_{d-1} & a_d & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \dots & b_e & 0 & 0 & \dots & 0 & 0 \\ 0 & b_0 & b_1 & \dots & b_e & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 0 & b_0 & b_1 & \dots & b_e \end{vmatrix} = 0.$$

To reiterate, f, g have a common factor iff $R(f,g) = 0$.

We can use this to have a similar result in projective. If we homogenize $f = a_dY^d + a_{d-1}Y^{d-1}Z + \cdots + a_0Z^d$ and likewise for g, then $\operatorname{res}(f,g) = 0$ iff f, g have a common zero in $\mathbb{P}^1$. (If $a_d = b_e = 0$, then the common zero is $(1 : 0) = \infty$.)

Furthermore, this resultant $R(f,g)$ can be seen as a "universal polynomial" in the coefficients of f and g: it still makes sense if the coefficients are in some commutative ring over k, in particular in $k[x_1, \dots, x_n]$. Considering then $f, g \in k[x_1, \dots, x_n][Y]$, we have $R(f,g) \in k[x_1, \dots, x_n]$ has the following property:

Property 14.1. For all $t_1, \dots, t_n \in k$, $R(f,g)(t_1, \dots, t_n) = 0$ iff $f(t_1, \dots, t_n, Y)$ and $g(t_1, \dots, t_n, Y)$ have a common root (as polynomials in Y).

Now, using this theory of resultants, we will prove our big theorem.

14.2 Proving Theorem 11.3

We'll set straight some notation. Let Y be a projective variety, X be any variety, $Z \subseteq X \times Y$ some closed subset. We wish to show that given the projection map $p_X : X \times Y \to X$, the image $p_X(Z)$ is closed.

Proof. First, we will prove the case for $Y = \mathbb{P}^1$ and $X = \mathbb{P}^m$. The next steps will be reductions of this step: after this, we will make X arbitrary, then we make Y arbitrary. For our initial case, we have $Z \subseteq \mathbb{P}^m \times \mathbb{P}^1$. Let the coordinates for $\mathbb{P}^m$ and $\mathbb{P}^1$ be $(x_0 : \cdots : x_m)$ and $(y_0 : y_1)$, respectively. $I(Z)$ is generated by bihomogeneous polynomials $F(X_0, \dots, X_m, Y_0, Y_1)$; we can write each F as

$$F = a_d(\underline{x}) \cdot Y_0^d + a_{d-1}(\underline{x}) \cdot Y_0^{d-1}Y_1 + \cdots + a_0(\underline{x}) \cdot Y_1^d,$$

where the coefficients a_i are all homogeneous of the same degree. Dehomogenize by Y_1, and look at $F' = F(X_0, \dots, X_m, Y, 1) \in k[X_0, \dots, X_m][Y]$. We have the following claim:

Claim 14.2. Let F, G be bihomogeneous as above, and dehomogenize them to F', G'. Then, $R(F', G')$ is a homogeneous polynomial in $k[X_0, \ldots, X_m]$ that vanishes on $p_X(Z)$.

This will give us one part of the proof. To complete the proof, we need to show the converse, in which case we have written $p_X(Z)$ as the vanishing locus of a set of homogeneous polynomials.

Proof. Suppose $x = (x_0 : \cdots : x_m) \in p_X(Z)$. Then, $\exists\, (y_0 : y_1) \in \mathbb{P}^1$ such that $(x, (y_0 : y_1)) \in Z$, i.e. $F(x_0, \ldots, x_m, y_0, y_1) = 0$ for all $F \in I(Z)$. We have two cases. If $y_1 \neq 0$, then y_0/y_1 is a common zero for all F', so $R(F', G')(x) = 0$. If $y_1 = 0$, then plugging in $y_1 = 0$ into F' gives us $a_d(x)Y_0^d = 0$, so x must be a common zero for all dominant coefficients $a_d(x)$. Looking at the last column of the resultant, we see that the resultant evaluates to 0, as desired. □

Now it suffices to prove the converse, i.e. if all resultants vanish at a point, then that point belongs in $p_X(Z)$. Suppose all $R(F', G')$ vanish at x, WLOG assume $x_0 \neq 0$. Again, we have cases. If x is a zero of all dominant coefficients a_d of the (F')'s, then the point $(x, (1 : 0)) \in Z$, so $x \in p_X(Z)$.

If x is not a common zero of all dominant coefficients, then there exists some $F_0 \in I(Z)$ such that its dominant coefficient F_0' is not zero at x. Consider $F_0'(x, Y, 1)$. This is a nonzero polynomial in one variable, so it vanishes at finitely many points $y_1, \ldots, y_n$.

Claim 14.3. At least one of $(x, (y_j : 1)) \in Z$.

Proof. Suppose none of $(x, (y_j : 1)) \in Z$, $1 \leq j \leq n$. Then $\forall\, j$, $\exists\, G_j \in I(Z)$ such that $G_j(x, (y_j : 1)) \neq 0$. Now, for every $(a_1, \ldots, a_n) \in k^n$, we can associate to it a polynomial

$$\sum_{i=1}^{n} a_i G_i(x, Y, 1).$$

Note that this is a linear combination of polynomials in $I(Z)$, hence itself in $I(Z)$. $F_0'(x, Y, 1)$ is also in $I(Z)$. By our inductive hypothesis, all $R(F', G')$ vanish at x for $F', G' \in I(Z)$, so the above sum has a common zero with $F_0'(x, Y, 1)$. The zeroes of the latter we wrote out as $y_1, \ldots, y_n$, so $\sum a_i G_i(x, Y, 1)$ vanishes at one of the y_j's. Consequently, the image of the morphism $k^n \to k^n$ given by

$$(a_1, \ldots, a_n) \mapsto \left(\sum_j a_j G_j, (x, y_1, 1), \ldots, \sum_j a_j G_j(x, y_n, 1) \right)$$

is contained in the union of the coordinate hyperplanes in k^n, as one of the components must be 0. But k^n is irreducible, so its image must be irreducible. This is a contradiction to the image being contained in the union of hyperplanes unless the image is contained in one of the coordinate hyperplanes. This is equivalent to saying that all G_j's vanish at the same $(x, y_i, 1)$, which contradicts our assumption that there is no point $(x, (y_j : 1))$ contained in Z. □

This tells us that $x \in p_X(Z)$, which immediately gives us our conclusion for the first step. Lucky for us, this is the hard part of the proof; the generalizations are not as bad.

For the second step, we still take $Y = \mathbb{P}^1$ but we now take X to be arbitrary. We can reduce this to the case where X is affine, so suppose $X \subseteq \mathbb{A}^m \subseteq \mathbb{P}^m$. We may cover X with open affines $X = \bigcup X_i$, $X_i \subseteq \mathbb{P}^m$. If $Z \subseteq X \times \mathbb{P}^1$ is closed, then for each i, $Z_i := Z \cap (X_i \times \mathbb{P}^1)$ is closed. But then we can invoke Step 1 to get $p(Z_i) \subseteq X_i \subseteq \mathbb{P}^m$ is closed, so the finite union $p(Z) = \bigcup_i p(Z_i) \subseteq X$ is also closed.

Finally, we take both X and Y to be arbitrary. It suffices to take $Y = \mathbb{P}^m$, since we always have $Y \subseteq \mathbb{P}^m$ for some m, and any closed subset of $X \times Y$ is closed in $X \times \mathbb{P}^m$. From here, the proof is not super easy, but also not difficult thanks to the blow-up machinery (wow sounds so violent) we developed last time.

Consider the blow-up $\widetilde{\mathbb{P}^n} = \mathrm{Bl}_p(\mathbb{P}^n)$. We will take the (lesser-used) projection map $q : \widetilde{\mathbb{P}^n} \to \mathbb{P}^{n-1}$, which we mentioned last time to be a locally trivial $\mathbb{P}^1$-bundle. We want to take advantage of this construction: if the $\mathbb{P}^1$-bundle was globally trivial, then we could use the second step to reach our conclusion by induction. Cover $\mathbb{P}^{n-1}$ by standard open affines U_i, $0 \leq i \leq n-1$. We have the following commutative diagram:

$$\begin{array}{ccccc}
X \times U_i \times \mathbb{P}^1 & \hookrightarrow & X \times \widetilde{\mathbb{P}^n} & \xrightarrow{1_X \times \pi} & X \times \mathbb{P}^n \\
\downarrow{\scriptstyle 1_X \times q} & & \downarrow{\scriptstyle 1_X \times q} & & \downarrow{\scriptstyle p_X} \\
X \times U_i & \hookrightarrow & X \times \mathbb{P}^{n-1} & \xrightarrow[p_X]{} & X
\end{array}$$

We can write the pullback of $X \times U_i$ as $X \times (U_i \times \mathbb{P}^1)$ in the top left because of an exercise from last time: $q^{-1}(U_i) \cong U_i \times \mathbb{P}^1$ since the projection is a locally trivial $\mathbb{P}^1$-bundle.

This gives us all the information we need. If $Z \subseteq X \times \mathbb{P}^n$ is closed, it follows that $(1_X \times \pi)^{-1}(Z)$ is closed. Restricting to $X \times U_i \times \mathbb{P}^1$, the intersection $(1_X \times \pi)^{-1}(Z) \cap (X \times U_i \times \mathbb{P}^1)$ is closed. We apply Step 2 to get the projection $(1_X \times q)((1_X \times \pi)^{-1}(Z) \cap (X \times U_i \times \mathbb{P}^1))$ is closed for all i (we are at the bottom left corner now). The union of all such i is the bottom center, given by $(1_X \times q)((1_X \times \pi)^{-1}(Z))$, which is closed. Finally, by our inductive hypothesis, p_X of this is closed, so we conclude. $\square$

15 03/25 - Degree of Maps (and Varieties)

15.1 Generically Finite Morphisms

This will help us define degree for a map, which in turn will help us define degree for a subvariety in projective space.

> **Definition 15.1** (Generically Finite). A morphism $f : X \to Y$ of (quasi-projective) varieties is called **generically finite** if $\exists \emptyset \neq U \subseteq Y$ open such that $f^{-1}(y)$ is finite for all $y \in U$.

The blow-up map is generically finite! For the open $U = Y \setminus \{0\}$, the map is bijective. But we are more interested in the following example:

Example 15.2 ("Main example" for generically finite)

Let $X \subseteq \mathbb{P}^n$ be a projective variety, $\mathbb{P}^{n-1} \cong H \subseteq \mathbb{P}^n$ a hyperplane, and $x_0 \in \mathbb{P}^n \setminus (X \cup H)$. Then, we will define a projection map $p : \mathbb{P}^n \setminus \{x_0\} \to H$ where, for $x \neq x_0$, $p(x)$ is the intersection of H with the line connecting x_0 and x (we will denote the line $\overline{x_0 x}$, so $p(x) = \overline{x_0 x} \cap H$.

We can pick coordinates such that $x_0 = (1 : 0 : \cdots : 0)$ and $H = (x_0 = 0)$. $\overline{x_0 x}$ is parametrically given by $(\lambda + \mu x_0 : \mu x_1 : \cdots : \mu x_n)$ where $\lambda, \mu \in k$. Thus, $p((x_0 : \cdots : x_n)) = (x_1 : \cdots : x_n)$. We will prove that this is generically finite below.

Claim 15.3. All of the fibers of $p|_X$ are finite.

Proof. Let $y \in p(X)$, so $f^{-1}(y) \subseteq \overline{x_0 x} = \mathbb{P}^1$. This is a closed subset. But note that $x_0 \notin f^{-1}(y)$, so the pre-image is not all of $\mathbb{P}^1$. But any closed strict subset of $\mathbb{P}^1$ is finite, voila. □

By Corollary 11.5, since X, H are projective, $p(X)$ is closed, hence a variety in $\mathbb{P}^{n-1}$. If $p(X) \subsetneq p^{n-1}$, we can repeat the process on $p(X)$ to eventually get a *surjective* morphism with finite fibers $f : X \to \mathbb{P}^m$ (so $f = p_{x_{n-m}} \circ \cdots \circ p_{x_1} \circ p_{x_0}$ is a composition of these projections). Even more, we can change coordinates such that the map is the projection map $(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_m)$.

Exercise 15.4. Interpret this as a unique projection from a higher dimensional linear subspace. (What I mean is, the initial map p is projecting from a point. If we compose two of these maps, this is equivalent to projecting from a line in $\mathbb{P}^n$. Generalize.)

15.2 Degree of Generically Finite Map

Now we define degree for a generically finite map. The following theorem gives us an algebraic interpretation of degree:

Theorem 15.5

Let $f : X \to Y$ be a dominant, generically finite morphism of varieties. Then, $K(Y) \hookrightarrow K(X)$ is a finite field extension.

In particular, $\operatorname{trdeg}_k K(Y) = \operatorname{trdeg}_k K(X)$ (i.e. $\dim Y = \dim X$).

Aside, related to Jarell's question in class: if you want a sneak peek on how the dimensions of the fibers behave...

Proposition 15.6

If $f : X \to Y$ is a morphism between two irreducible affine varieties over an algebraically closed field k, then the function that assigns to each point of X the dimension of the fiber it belongs to is upper semicontinuous on X.

Remark 15.7. The dominant condition here is nothing to worry about: we can just consider p as a map from X to $p(X)$.

The degree of the map is now very natural to define.

Definition 15.8 (Degree of map)**.** The **degree** of the map $f : X \to Y$ is the degree of the field extension $[K(X) : K(Y)]$.

So you can read a birational map from its induced field extension. A birational map is 1-1 on almost all points, which means the two function fields are the same.

We add the following statement to Theorem 15.5.

Theorem 15.9 (Adding on to Theorem 15.5)

If $\operatorname{char} k = 0$, then $\deg f$ is the same as $\# f^{-1}(y)$ for $y \in U \subseteq Y$ non-empty open.

Proof. WLOG we may assume X, Y are affine. (Check this! Left as exercise.) We can also factor $f : X \xhookrightarrow{\Gamma_f} X \times Y \xrightarrow{p_2}$, with $X \times Y \subseteq \mathbb{A}^{n+m}$, $Y \subseteq \mathbb{A}^m$, and p_2 being the projection $\mathbb{A}^{n+m} \to \mathbb{A}^m$. We may decompose this into n projections, one variable at a time. Thus, we may assume $X \subseteq \mathbb{A}^{m+1}$ and $Y \subseteq \mathbb{A}^m$, with projection $p : \mathbb{A}^{m+1} \to \mathbb{A}^m$ being the one removing the first component.

Given our map $X \to Y$, we have a pullback map $A(Y) \to A(X) = A(Y)[x_0]$, so $K(X) = K(Y)(x_0)$. We now prove the following:

Claim 15.10. x_0 is algebraic over $K(Y)$.

Proof. Assume by contradiction that x_0 is transcendental over $K(Y)$. Take $F \in I(X) \subset k[x_0, \ldots, x_m]$. We may write $F = F_0(x_1, \ldots, x_m) \cdot x_0^d + F_{d-1}(x_1, \ldots, x_{m-1}) \cdot x_0^{d-1} + \ldots$. Since x_0 is transcendental, we must have $F_i \equiv 0$ on Y. But then this means if $(x_1, \ldots, x_n) \in Y$, then $\forall\, x_0 \in k$, $(x_0, x_1, \ldots, x_m) \in X$, so $f^{-1}(y) = \mathbb{A}^1$. But this is a contradiction to the finiteness of $f^{-1}(y)$, so x_0 is algebraic over $K(Y)$. □

It is immediate from here that $K(Y) \subseteq K(X)$ is an algebraic, and hence finite, extension.

Now we prove the second part of the theorem (written as Theorem 15.9). Assume now $\operatorname{char} k = 0$. Let F be the minimal polynomial of x_0 over $K(Y)$. (We can assume coefficients of F are in $A(Y)$ by clearing denominators.) Denote $d := \deg(f)$. Take $\Delta(x_1, \ldots, x_m)$ to be the discriminant of F. Since F is irreducible with coefficients in $K(Y)$ and $\operatorname{char} k = 0$, we have $\Delta \not\equiv 0$ on Y. This means that $(\Delta = 0)$ and $(F_d = 0)$ are proper closed subsets of Y, hence every fiber for any $y \in U = Y \setminus ((\Delta = 0) \cup (F_d = 0))$ has d points. brush this up $\square$

Remark 15.11. The same proof works if $K(Y) \hookrightarrow K(X)$ is a **separable** extension in char $k = p > 0$. (Recall that separable means that $f' \not\equiv 0$ if f is the minimal polynomial of any $x_0 \in K(X)$.)

On defining separable extension:

> *"You should have learned about separable extensions in any field theory class. If not, that professor should be fired." -Popa*

15.3 Degree of Projective Variety

As one may expect, it uses the definition of degree for a generically finite map.

Definition 15.12 (Degree)**.** Let $X \subseteq \mathbb{P}^n$ be a projective variety. Take a surjective projection map $f : X \to \mathbb{P}^m$, $m = \dim X$. This is a generically finite map, so it has a well-defined degree. We define the **degree of X** (in $\mathbb{P}^n$) to be $\deg(X) := \deg(f)$.

This seems to be good, but someone should call Houston because we have a problem. This definition is dependent on the choice of our projection f! Or at least, it seems like it for now. It turns out (thankfully) that the degree of X is actually independent of our choice of f, but we will *assume this for now* and prove it later.

Remark 15.13. Although $\deg(X)$ doesn't depend on f, it does depend on the embedding into $\mathbb{P}^n$. For instance, we can embed $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ as either a line or a conic, which gives degrees 1 and 2, respectively.

Remark 15.14. In general, $\deg(X)$ is the number of points in the intersection of X with a general $(n-m)$-dimensional linear subspace of $\mathbb{P}^n$, which in turn is just equal to $\#(X \cap H_1 \cap \cdots \cap H_m)$ where each H_i is a general hyperplane in $\mathbb{P}^n$.

Example 15.15

If $X = Z(F) \subseteq \mathbb{P}^n$ is a hypersurface, then $\deg(X) = \deg(F)$. (Whew!) The proof of this is reminiscent of our proof for Bezout's Theorem that we did in $\mathbb{P}^2$, which we can just generalize to $\mathbb{P}^n$.

Example 15.16

Take the conic $X = Z(xy - z^2) \subseteq \mathbb{P}^2$, and consider the map $X \to \mathbb{P}^1$ induced by the projection map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $(x : y : z) \mapsto (x : y)$. If $\operatorname{char} k \neq 2$, then the fibers consist of two points except over $(0 : 1)$ and $(1 : 0)$. (You're finding solutions to $z^2 = a$.) But if $\operatorname{char} k = 2$, then everything is totally messed up! In this case, $z^2 = a$ always has one unique solution, so all fibers consist of just one point. The failure is due to the fact that $f' = 0$, so the field extension $K(x) \hookrightarrow K(z)$, $x \mapsto z^2$ is inseparable.

16 03/27 - Morphisms of Projective Varieties

Theorem 16.1

Let $X, Y \subseteq \mathbb{P}^n$ be quasi-projective varieties. Then,

1. If $X \cap Y \neq \emptyset$, then every irreducible components of $X \cap Y$ has dimension $\geq \dim X + \dim Y - n$.
2. If X, Y is closed and $\dim X + \dim Y \geq n$, then $X \cap Y \neq \emptyset$.

Proof. Consider the cone construction $C^0(X) := \pi^{-1}(X)$, where $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$.

Exercise 16.2. Prove that the cone is quasi-projective, irreducible if X is irreducible, and $\dim C^0(X) = \dim X + 1$.

Then let $C(X) = \overline{C^0(X)} \subseteq \mathbb{A}^{n+1}$. In particular, if X is closed, then $C(X)$ is just the affine cone over X ($C(Z) = C^0(Z) \cup \{0\}$). We now prove (1). We have

$$\begin{aligned} C^0(X \cap Y) &= C^0(X) \cap C^0(Y) \\ &= (C^0(X) \times C^0(Y)) \cap \Delta_{\mathbb{A}^{2n+2}} \\ \implies \dim(C^0(X) \cap C^0(Y)) &\geq \dim C^0(X) + \dim C^0(Y) - (n+1) \\ \implies \dim(X \cap Y) + 1 &\geq (\dim X + 1) + (\dim Y + 1) - (n+1) \\ &= \dim X + \dim Y - n + 1, \end{aligned}$$

as desired. (We subtract $(n+1)$ in the third line because $\Delta_{\mathbb{A}^{2n+2}}$ is carved out by $n+1$ linear equations.)

Now we prove (2). The same reasoning as in (1) shows that each component of $C(X) \cap C(Y)$ has dimension $\geq \dim X + \dim Y - n + 1$. By hypothesis, this is ≥ 1. Thus, $C(X) \cap C(Y) = C(X \cap Y) = \overline{C^0(X \cap Y)}$ has to contain some nonzero points, so $C^0(X \cap Y) \neq \emptyset$. $\square$

16.1 Dimension of Fibers of Morphism

We now prefer a very important theorem on the dimensions of the fibers of a morphism. We won't prove it in class, since the proof is involved and not enlightening, but we will look at some of its applications. (Hopefully) Accessible proofs can be found in Harris (Thm 11.12) and Shafarevich (somewhere idk).

Definition 16.3 (Upper Semicontinuous Function)**.** Let X be a variety, and $\varphi : X \to \mathbb{Z}$ a function. Then, φ is **upper semicontinuous** if $\forall\, k \in \mathbb{Z}$, the set $\{x \in X \mid \varphi(x) \geq k\}$ is closed.

Consequently, if φ is upper semicontinuous, then there exists some open $\emptyset \neq U \subseteq X$ on which φ attains its minimal value.

This seems like a very odd definition that really comes out of nowhere, but it is significant because so many things in algebraic geometry turn out to be upper semicontinuous. For example, the rank of stalks of a coherent sheaf on X is given by such a function.

Definition 16.4 (Dimension w.r.t point)**.** If Z is an algebraic set containing x, then $\dim_x Z$ is the maximal dimension among components passing through x.

We now can state our theorem.

Theorem 16.5

Let X be a variety and $f : X \to \mathbb{P}^n$ a morphism. For each $x \in X$, denote $X_x := f^{-1}(f(x))$. Then, the function $\varphi : X \to \mathbb{N}$, $\varphi(x) = \dim_x X_x$ is upper semicontinuous. Moreover, $\dim X = \dim \overline{f(X)} + \varphi_0$, where $\varphi_0 := \min_{x \in X} \varphi(x)$.

Remark 16.6. This is really a statement about any $f : X \to Y$ between quasi-projective varieties.

Corollary 16.7

Let $f : X \to \mathbb{P}^n$ be a morphism, X quasi-projective. Denote $Y = \overline{f(X)}$. Then,

1. $\forall\, y \in f(X)$, every irreducible component of $f^{-1}(y)$ has dimension $\geq \dim X - \dim Y$.
2. $\exists\, \emptyset \neq U \subseteq Y$ oepn such that $\forall\, y \in U$, $\dim f^{-1}(y) = \dim X - \dim Y$.

Remark 16.8. There is a very surprising statement embedded in the second part of the theorem. Y is the closure of the image, but we know very little about the image of a morphism! At best, we know that it is a constructible set. Then why must there exist an open in Y? This is actually not necessarily true a priori, but the above corollary guarantees this fact. We write it explicitly below.

Corollary 16.9

The image of a dominant morphism of quasi-projective varieties contains a nonempty open set.

Corollary 16.10

Let $f : X \to Y$ be a morphism of quasi-projective varieties. Then, the image of every constructible set is constructible.

This generalizes (and hence implies) Chevalley's Theorem, at least the version that we stated without the information about the dimension of the fibers. (Theorem 11.18)

Proof. We induct on $\dim Y$. Note that we can always decompose X into finitely many irreducibles, and if the images of each irreducible is constructible, then the finite union of the images is also constructible. Thus, we may assume X is irreducible and f dominant.

By Corollary 16.9 above, $\exists \emptyset \neq U \subseteq f(X)$. Denote $Z = Y \setminus U \subsetneq Y$ closed. We have $\dim Z < \dim Y$. We can write

$$f(X) = U \cup f(f^{-1}(Z)).$$

$f(f^{-1}(Z))$ is constructible by induction, and U is open, so their union is constructible. The conclusion follows. $\square$

We now prove the first corollary (16.7).

Proof. We will prove when $f : X \to Y$ is a morphism of projective varieties. (Because it is between projective varieties, it is surjective.) check this For $r \in \mathbb{N}$, denote

$$X(r) := \{x \in X \mid \dim_x X_x \geq r\}.$$

Recall $X_x = f^{-1}(f(x))$. Fix $y \in Y$, and let X' be an irreducible component in $f^{-1}(y)$. Fix $x \in X'$ not in any other component. We then have

$$\dim X' = \dim_x X_x \geq \varphi_0 = \dim X - \dim Y,$$

where the last equality follows from Theorem 16.5. This gives us (1).

For (2), we wish to show $\forall Z \subseteq \varphi^{-1}(\varphi_0 + 1)$ irreducible component, $f(Z) \subsetneq Y$. Let $x \in Z$ but not in some other component of $\varphi^{-1}(\varphi_0 + 1)$. Then, there exists an irreducible component X' of X_x passing through x such that $\dim X' \geq \varphi_0 + 1$. Thus, by our choice of x, we get $X' \subseteq Z \subsetneq X$. Then, for $f|_Z : Z \to f(Z)$, the function φ has minimum value $\geq \varphi_0 + 1$. Applying the theorem for $f|_Z$, we get

$$\dim \overline{f(Z)} \leq \dim Z - \varphi_0 - 1 \leq \dim X - \varphi_0 - 2 = \dim Y - 2.$$

This completes the proof. go over this $\square$

Remark 16.11. This highlights the fact that many things we're dealing with are parameterized by Zariski closed subsets. explain more

"We can apply induction to anything that's bad in life." -Popa

16.2 Grassmannians

This is a fundamental construction that we'll use frequently moving forward. We first define the Grassmannian as a set. Define

$$G(k,n) := \{k - \dim \text{ linear subspaces of } K^n\}.$$

Let V be a K-vector space of dimension n. Then, we may define

$$G(k,V) = \{k - \dim \text{ linear subspaces of } V\}.$$

Note that these two coincide once we choose a basis for V. We can also consider subspaces in $\mathbb{P}^n$; denote

$$G(k,n) = \mathbb{G}(k-1,n-1) = \{(k-1) - \dim \text{ linear subspaces of } \mathbb{P}^{n-1}\},$$

and similarly for $G(k,V) = \mathbb{G}(k-1,\mathbb{P}(V))$.

Fact 16.12. $G(k,n)$ is a **projective variety** of dimension $k(n-k)$.

Example 16.13

Observe $G(1,n) \cong \mathbb{P}^{n-1}$.

Understanding the Grassmannian as a projective variety is not entirely clear a priori. This is achieved through the **Plücker embedding**

$$G(k,V) \hookrightarrow \mathbb{P}(\bigwedge^k V).$$

So the key to understanding Grassmannians is through exterior algebra. Let us define the Plücker embedding.

Suppose $W \subsetneq V$ is a k-dimensional vector subspace, choose basis $\langle v_1,\ldots,v_k\rangle$. Then, $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$, so $[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V)$. (finish this)

h

17 04/03 - Exterior Algebra

Exterior algebra is perhaps the most righteous way to learn about Grassmannians, and it is an important topic that is often overlooked in algebra classes (I speak from personal experience).

> **Author's Note 17.1.** I wasn't able to make it to class today, so these are based off of Eliot's notes. Thanks Eliot!

Assume $\operatorname{char} k \neq 2$. Let V be an n-dimensional vector space over k, and choose some basis $e_1, \ldots, e_n$ of V. The exterior algebra of V,denoted E, is defined in the following way:

$$E = \bigwedge V := \bigwedge^0 V \oplus \bigwedge^1 V \oplus \cdots \oplus \bigwedge^n V,$$

where

$$\bigwedge^k V = \bigotimes_{i=1}^{k} V / (v_1 \otimes \cdots \otimes v_k - \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}).$$

In other words, $\bigwedge V = \bigotimes V / \langle x \otimes y + y \otimes x \rangle$, where $\langle x \otimes y + y \otimes x \rangle$ denotes the relations generated by $x \otimes y + y \otimes x$ for all $x, y \in V$. Therefore, for all $x, y \in V$, $x \wedge y = -y \wedge x$ by construction.

Note that E is a graded ring by constructino, with ring structure given by $\wedge : E \times E \to E$ taking $(v, w) \mapsto v \wedge w$. Because we assumed $\operatorname{char} k \neq 2$, we have $x \wedge x = -x \wedge x \implies x \wedge x = 0$ for all $x \in E$.

The exterior algebra $E = \bigwedge V$ satisfies the following properties:

1. For all $\lambda_1, \lambda_2 \in k$ and $v_1, v_2, w \in E$,

$$(\lambda_1 v_1 + \lambda_2 v_2) \wedge w = \lambda_1 (v_1 \wedge w) + \lambda_2 (v_2 \wedge w),$$

2. $v_1 \wedge v_2 = -v_2 \wedge v_1$ (we call the operation **skew-commutative**),

3. $\wedge$ is associative.

Let's work a little more tangibly with these wedge products. An element in $\bigwedge^k V$ is expressed as

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1, \ldots, i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Note that $a_{i_1, \ldots, i_k} \in k$ and that the $e_{i_1} \wedge \cdots \wedge e_{i_k}$ are linearly independent over k. The set of all $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ forms a basis for $\bigwedge^k V$, so $\dim \bigwedge^k V = \binom{n}{k}$. In particular, $\bigwedge^n V \cong k e_1 \wedge \cdots \wedge e_n$, since $\dim \bigwedge^n V = 1$.

17.1 Exterior Product Properties

We continue with understanding the exterior product. Suppose $\varphi : V \to V$ is a linear transformation. If $e_1, \ldots, e_n$ is a basis for V, then φ can be identified with a matrix $A \in M_n(k)$. This induces a map $\bigwedge^k \varphi : \bigwedge^k V \to \bigwedge^k V$, and in particular a map $\bigwedge^n \varphi : \bigwedge^n V \cong k \to \bigwedge^n V \cong k$. (To clarify, there are two k's here: one for the field over which V is a vector space, and one for the power of the exterior algebra.)

Exercise 17.2. $\bigwedge^n \varphi$ corresponds to multiplication by $\det(A)$.

As a consequence of the above, $\bigwedge^n \varphi = 0$ tells us that $\operatorname{rank} \varphi < n$; these are equivalent statements.

Corollary 17.3

Suppose $v_1, \ldots, v_k$ are linearly independent, and suppose $v_1', \ldots, v_k' \in W := \langle v_1, \ldots, v_k \rangle$. Write $v_i' = \sum_j a_{ij} v_j$. Then,

$$v_1' \wedge \cdots \wedge v_k' = \det(a_{ij})_{i,j} v_1 \wedge \cdots \wedge v_k.$$

Similar to how in tensor algebra, we often focus our attention on pure tensors (as any element is expressible as a sum of pure tensors), we want to investigate "pure" elements in our exterior algebra. We define two types of decompositions:

Definition 17.4 (Completely/partially decomposable)**.** An element $\omega \in \bigwedge^k V$ is **completely decomposable** (a "**pure wedge**") if $\omega = v_1 \wedge \cdots \wedge v_k$ for $v_1, \ldots, v_k \in V$. $\omega \in \bigwedge^k V$ is **partially decomposable** if there exists $v \in V$ and $u \in \bigwedge^{k-1} V$ such that $\omega = v \wedge u$.

Proposition 17.5

Let $\omega \in \bigwedge^k V$. Then,

1. If ω is partially decomposable, then $\omega \wedge \omega = 0$,
2. If ω is partially decomposable, then the linear transformation $\phi_\omega : V \to \bigwedge^{k+1} V$ taking $v \mapsto v \wedge w$ has nonzero kernel,
3. If $v_1, \ldots, v_m$ forms a basis for $\ker \phi_\omega$, then $\omega = v_1 \wedge \cdots \wedge v_m \wedge \eta$ for some $\eta \in \bigwedge^{k-m} V$,
4. ω is completely decomposable iff $\dim \ker \phi_\omega = k$.

Proof. **Part (1):** Write $\omega = v \wedge u$ for some $v \in V$, $u \in \bigwedge^{k-1} V$. Then, $\omega \wedge \omega = (v \wedge u) \wedge (v \wedge u) = -(v \wedge v) \wedge (u \wedge u) = 0$.

Both **(2)** and **(4)** are special cases of (3) (these implications are left as an exercise), so it remains to prove (3).

Part (3): Complete $v_1, \ldots, v_m$ to a basis $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ of V. Write

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k}$$

and consider $v_j \wedge \omega$. Using properties of the wedge product, one can see that

$$v_j \wedge \omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_k}.$$

$v_j \wedge v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$ if $j = i_r$ for some r, otherwise it is a basis vector for $\bigwedge^{k+1} V$. Hence, $v_j \wedge \omega = 0$ iff $a_{i_1 \ldots i_k} = 0$ for every $\{i_1, \ldots, i_k\} \not\ni j$. Because $v_1, \ldots, v_m$ are a basis for $\ker \phi_\omega$, we know $v_1 \wedge \omega = \cdots = v_m \wedge \omega = 0$, implying that $a_{i_1 \ldots i_k} = 0$ if $\{1, \ldots, m\} \not\subset \{i_1, \ldots, i_k\}$. □

17.2 Applications to Grassmannians

For an n-dimensional vector space V over k, recall that $G(k, V)$ denotes the set of k-dimensional linear subspaces of V. Alternatively, $\mathbb{G}(k-1, \mathbb{P}(V))$ denotes the set of $(k-1)$-planes in $\mathbb{P}(V)$.

Lemma 17.6

(Homework) There exists an injection $\iota : G(k, V) \hookrightarrow \mathbb{P}(\bigwedge^k V)$. Given $W = \langle v_1, \ldots, v_k \rangle \in G(k, V)$, this map takes $W \mapsto [v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V) \cong \mathbb{P}^{\binom{n}{k}-1}$.

Proof. The lemma follows from showing that the map is well-defined and injective. The details should be done in the corresponding problem on the homework. □

The reason why we introduced exterior algebras is for this: the above embedding realizes the Grassmannian as a projective subvariety of $\mathbb{P}(\bigwedge^k V)$.

Lemma 17.7

Moreover, ι realizes $G(k, V)$ as a closed subset in $\mathbb{P}(\bigwedge^k V)$. We call ι the **Plücker embedding**.

Proof. By Proposition 17.5, $[\omega] \in \mathrm{Im}(\iota)$ if and only if $\phi_\omega : V \to \bigwedge^{k+1} V$ has rank $\le (n-k)$. But this is true iff all the $(n-k+1) \times (n-k+1)$ minors of ϕ_ω are 0. This gives polynomial equations carving $\mathrm{Im}(\iota) \subset \mathbb{P}(\bigwedge^k V)$.

More specifically, fix a basis $e_1, \ldots, e_n \in V$. The coordinates on $\mathbb{P}(\bigwedge^k V)$ are $[\cdots : x_{i_1 \ldots i_k} : \ldots]_{1 \le i_1 < \cdots < i_k \le n}$. Write $\omega \in \bigwedge^k V$ as $\omega = v_1 \wedge \cdots \wedge v_k$. Then, $\ker \phi_\omega = \langle v_1, \ldots, v_k \rangle$, where $v_i = \sum_j x_{ij} e_j$. Now, recall that ϕ_ω takes $e_1 \mapsto e_1 \wedge \omega$. We have

$$\begin{aligned} e_1 \wedge \omega &= e_1 \wedge v_1 \wedge \cdots \wedge v_k \\ &= e_1 \wedge \left(\sum x_{1j} e_j\right) \wedge \cdots \wedge \left(\sum x_{kj} e_j\right) \\ &= \sum x_{1i_1} \cdots x_{ki_k} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned}$$

Similar calculations can be performed for the other e_i's. Each $x_I = x_{1i_1} \cdots x_{ki_k}$ corresponds to $e_{i_1} \wedge \cdots \wedge e_{i_k}$. The minors of ϕ_ω give the equations for the Grassmannian in the x_I. $\square$

Let's put this into practice.

Example 17.8

Suppose $k = 2$. Then, the embedding we are working with looks like $\iota : G(2, V) \hookrightarrow \mathbb{P}(\bigwedge^2 V)$. We claim that $\mathrm{Im}(\iota)$ is given by the equations $\omega \wedge \omega = 0$. (This is a quadratic relation in the x_I's.) These are called the **Plücker relations**. The proof is left as an exercise for the reader.

Example 17.9 ($G(2,4)$)

The first nontrivial Grassmannian is for $(k, n) = (2, 4)$. We have $G(2, 4) = \mathbb{G}(1, 3) \hookrightarrow \mathbb{P}(\bigwedge^2 k^4) = \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$. (Recall $G(2, 3) \cong G(1, 3) \cong \mathbb{P}^2$.) The dimension of $G(k, n)$ is $k(n-k)$, so $\dim G(2, 4) = 4$. Therefore, $G(2, 4)$ is a hypersurface in $\mathbb{P}^5$, i.e. a quadric. Let e_1, e_2, e_3, e_4 be a basis for k^4. We have that

$$\omega = \sum_{1 \le i < j \le 4} x_{ij} e_i \wedge e_j = x_{12}(e_1 \wedge e_2) + \cdots + x_{34}(e_3 \wedge e_4),$$

so $x_{12}, \ldots, x_{34}$ are our coordinates on $\mathbb{P}^5$. We also see that $\omega \wedge \omega = 0$ iff $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$. This is the equation of $G(2, 4)$ in $\mathbb{P}^5$.

Remark 17.10. Our observations here are not especially nice. In fact, every $G(k, n)$ is defined by quadrics. (For a reference, look at the section on Grassmannians in Harris's book.)

18 04/05 - Schubert Cycles

18.1 Defining Schubert Cycles

[insert stuff here cuz i was late] (schubert)

So, if we fix a basis $w_1, \ldots, w_m$ for L and compete it to a basis $w_1, \ldots, w_m, w_{m+1}, \ldots$ of V, we can look at

$$\omega \wedge w_{i_1} \wedge \cdots \wedge w_{i_{m-\ell+1}} = 0$$

for $1 \le i_1 < \cdots < i_{m-\ell+1} \le m$. Expand ω in terms of the basis; then, these wedge products are equations in the coefficients of ω, and in particular these give the equations for $\sum_\ell(L)$.

Consider $L \subseteq V$ where $m = \dim L = n - k$, and take $\ell = 1$. Then, we have

$$\sum_1(L) := \{W \in G(k, V) \mid \dim(W \cap L_{\geq} 1\}$$
$$= \{[\omega] \in \mathbb{P}(\bigwedge^k V) \mid \omega \wedge v_1 \wedge \cdots \wedge v_{n-k} = 0 \,\forall\, v_1, \ldots, v_{n-k} \in L\}.$$

Claim 18.1. $\sum_1(L) = G(k, V) \cap H$ for a hyperplane H.

Proof. Let $v_1, \ldots, v_{n-k}$ be a basis for L, and extend it by $v_{n-k+1}, \ldots, v_n$ to a basis for V. Denote

$$\eta = v_1 \wedge \cdots \wedge v_{n-k} \in \bigwedge^{n-k} V$$
$$H_\eta = \{[\omega] \in \mathbb{P}(\bigwedge^k V) \mid \omega \wedge \eta = 0\}.$$

Now for the punchline.

Exercise 18.2. $\sum_1(L) = \{[\omega] \in G(k, V) \mid \omega \wedge \eta = 0\}$; in other words, $\sum_1(L) = G(k, V) \cap H_\eta$.

It thus suffices to prove that H_η is a hyperplane. We can express ω as

$$\omega = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1 \cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k},$$

as the collection of all $v_{i_1} \wedge \cdots \wedge v_{i_k}$ forms a basis for $\bigwedge^k V$. Then, $\omega \wedge \eta = 0$ is equivalent to

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1 \cdots i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge \underbrace{v_1 \wedge \cdots \wedge v_{n-k}}_{\eta} = 0.$$

But if any $i_r \in \{1, \ldots, n-k\}$, then the product vanishes. Thus, the only possibly nonzero term in the sum is when $\{i_1, \ldots, i_k\} = \{n - k + 1, \ldots, n\}$. This wedge product may be nonzero, so we require the coefficient to be 0. Thus, the equation of H_η is given by $x_{n-k+1,\ldots,n} = 0$, so it is a hyperplane. $\square$

18.2 Properties of the Grassmannian

Our motivation for introducing these Schubert cycles is to give us new information about Grassmannians. We glean the rewards now.

Corollary 18.3

$G(k, V) \setminus \sum_1(L)$ is affine.

Proof. This follows from the observation that $\mathbb{P}(\bigwedge^k V) \setminus H_\eta = \mathbb{A}^{\binom{n}{k}-1}$ is affine. $\square$

We can do even better by specifying the dimension.

Proposition 18.4

$G(k,V) \setminus \sum_1(L) \cong \mathbb{A}^{k(n-k)}$.

This is incredible, because this tells us many things about the Grassmannians:

- $G(k,V)$ is irreducible,
- $\dim G(k,V) = k(n-k)$,
- $G(k,V)$ is rational,
- $G(k,V)$ is smooth.

These show that Grassmannians are really a very special kind of projective variety, one that algebraic geometers love to work with due to all of the properties it satisfies. Our life is hinging on Proposition 18.4, though, so let's prove it.

Proof. Observe $G(k,V) \setminus \sum_1(L) = \{W \in G(k,V) \mid W \cap L = \{0\}\}$. Fix bases $v_1, \ldots, v_{n-k}$ for L and $B = \{v_1, \ldots, v_{n-k}, v_{n-k+1}, \ldots, v_n\}$ for V. For any $W \in G(k,V)$, we can write $W = \langle w_1, \ldots, w_k \rangle$. Express each w_j in terms of our basis B. Then, via Gaussian elimination, the matrix for $\{w_1, \ldots, w_k\}$ up to coordinate change looks like

$$\left(\begin{array}{ccc|ccc} a_{1,1} & \cdots & a_{1,n-k} & 1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ a_{k,1} & \cdots & a_{k,n-k} & & . & 1 \end{array}\right)$$

where the empty spaces on the right are all 0 and the left is just some $k \times (n-k)$ matrix (a_{ij}). This gives a correspondence

$$W \to A \in M_{k,n-k} \cong \mathbb{A}^{k(n-k)}.$$

Exercise 18.5. Check that the above is in fact a bijection.

The finish line is in sight now. We have

$$G(k,V) \setminus \sum_1 (L) = \{W \mid W \oplus L = V\} \cong \operatorname{Hom}_k(V/L, L) \cong \mathbb{A}^{k(n-k)},$$

as desired. □

18.3 Incidence Correspondences and Dimension Calculations

A good reference for incidence correspondence is Harris's book, for those who would like to read more.

Let $\dim L = m$. Recall $\sum_\ell(L) = \{W \in G(k,V) \mid \dim(W \cap L) \geq \ell\}$. Define

$$\mathcal{L} = \{(U,W) \mid U \subseteq W\} \subseteq G(\ell, L) \times G(k,V).$$

We have projection maps $p_1 : \mathcal{L} \to G(\ell, L)$ and $p_2 : \mathcal{L} \to G(k,V)$. Note that, by construction, $\operatorname{Im} p_2 = \sum_\ell(L)$. We see that p_1 is surjective and $p_1^{-1}(U) = G(k-\ell, V/U)$ (check this) which is irreducible of dimension $(k-\ell)(n-k)$.

> **Exercise 18.6.** (Homework) If $f : X \to Y$ is surjective, Y is irreducible, and the fibers are all irreducible of the same dimension d, then X is irreducible. (Additionally, $\dim X = \dim Y + d$.)

Invoking this, we get that $\mathcal{L}$ is an irreducible closed subset in $G(\ell, L) \times G(k,V)$ (hence a **projective variety**) of dimension

$$\begin{aligned}\dim \mathcal{L} &= \dim G(\ell, L) + \dim p_1^{-1}(U) \\ &= \ell(m-\ell) + (k-\ell)(n-k).\end{aligned}$$

We can relate this back to our Schubert cycles: we have $\operatorname{Im} p_2 = \sum_\ell(L)$, so $\sum_\ell(L)$ is irreducible as well.

We still have some work to do to determine the dimension of this Schubert cycle, though. This requires looking at the dimensions the fibers of our projections. Given any $W \in G(k,V)$, the only possible subspace in the preimage is $p_2^{-1}(W)$ is $W \cap L = U$. (check).

Theorem 18.7

$\dim Z = \dim p_2(Z) + \dim p_2^{-1}(W)$.

p_2 is birational, so we have $\dim \sum_\ell(L) = \dim Z = \ell(m-\ell)+(k-\ell)(n-k)$. For the specific case $m = n-k$ and $\ell = 1$, then we get the identity $(n-k-1)+(k-1)(n-k) = k(n-k)-1$! Obviously you could verify this yourself by expanding, but it's cool that algebraic geometry produces arithmetic equations like this.

Example 18.8

Consider $G(2,4) \subseteq \mathbb{P}^5$; this is a quadric hypersurface. We can associate this to $\mathbb{G}(1,3)$, the lines in $\mathbb{P}^3$. Consider $Q = \ell \mid \ell \cap \ell_1 \neq \emptyset, \ell \cap \ell_2 \neq \emptyset\} \subseteq \mathbb{G}(1,3)$, where ℓ_1, ℓ_2 are two fixed lines in $\mathbb{P}^3$. We have two cases depending on if the lines intersect.

Suppose $\ell_1 \cap \ell_2 = \emptyset$, i.e. they are skew lines. The line $\ell_1 \subseteq \mathbb{P}^3$ corresponds to a plane $L_1 \subseteq k^4$, and under this correspondence we can write $\{\ell \mid \ell \cap \ell_1 \neq \emptyset\} = \sum_1(L_1)$. Likewise, we can construct $\sum_2(L_2)$. By definition, $Q = \sum_1(L_1) \cap \sum_2(L_2) = G(2,4) \cap H_1 \cap H_2 \subseteq \mathbb{P}^5$.

What is going on here? We start with $G(2,4)$, a quadric in $\mathbb{P}^5$. We cut it with a hyperplane, so it is a quadric in $\mathbb{P}^4$, and cutting it by another hyperplane identifies it as a quadric in $\mathbb{P}^3$. But stepping back, any line intersecting ℓ_1 and ℓ_2 is determined by choosing a point on ℓ_1 and ℓ_2, respectively. Noting $\ell_1, \ell_2 \cong \mathbb{P}^1$, this gives us a new proof that $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Neat!

Exercise 18.9. Consider the case where $\ell_1 \cap \ell_2 \neq \emptyset$ next. Identify this scenario with an degenerate version of the quadric, and convince yourself that this makes sense.

19 04/17 - Tangent Cone

Remark 19.1. (1) Think of Taylor expansions as trying to "localize" X at x. (2) $TC_pX \subsetneq T_pX$, and T_pX is defined by terms f_1, $\forall f \in I(X)$.

Example 19.2

Missed the first one, but

1. Let $X = (y^2 = x^3 + x^2) \subsetneq \mathbb{A}^2$. Then, $TC_0X = (y^2 = x^2) \subsetneq T_0X = \mathbb{A}^2$.
2. Let $X = (y^2 = x^3) \subsetneq \mathbb{A}^2$. Then, $TC_0X = (y^2 = 0) \subsetneq T_0X = \mathbb{A}^2$ is the double line.

We have a really nice geometric interpretation of the projective space of TC_xX: $\mathbb{P}TC_xX$ is the exceptional divisor of the blow-up of X at x.

Example 19.3

Let $C \subsetneq \mathbb{A}^2$, and write $f(x_0, x_1) = \sum a_{ij}x_0^ix_1^j = a_{00} + a_{10}x_0 + a_{01}x_1 + \dots$. Suppose $a_{00} = 0$, i.e. $0 \in C$. The tangent line at 0 is given by $a_{10}x_0 + a_{01}x_1$ if $(a_{10}, a_{01} \neq (0,0)$. Let the coordinates on $\mathbb{P}^1$ be $(y_0 : y_1)$. By direct calculation, the proper transform $\widetilde{C}$ of C is given by

$$a_{10}y_0 + a_{01}y_1 + a_{11}y_0x_1 + a_{20}x_0y_0 + a_{02}x_1y_1 + \dots = 0.$$

Over the origin, $(x_0, x_1) = (0,0)$, in which we get our tangent line $a_{10}y_0 + a_{01}y_1 = 0$.

Exercise 19.4. (Homework) Let $P = \{0\} \in X \subsetneq \mathbb{A}^n$. Consider the blow-up $\pi : \widetilde{X} \to X$; note $\widetilde{X} \subsetneq \mathrm{Bl}_0(\mathbb{A}^n)$ and $X \subsetneq \mathbb{A}^n$. Then, the exceptional divisor $E \simeq \mathbb{P}TC_0X$.

Example 19.5

Let $X = (x^2 + y^2 + z^2 = 0) \subseteq \mathbb{A}^3$. (We could replace 2 with any m; the $m = 2$ case is called the *quadric cone.*)

uh oh here come diagrams i can't draw, but you start with a cone X with vertex at 0. The preimage of X under μ is a shape resembling a cylinder (?) as the projectivizied tangent cone, and the cross-section along the middle is $\mathbb{P}^2 = \operatorname{Exc}(\mu)$. We have $f = f_2 = f^{\text{in}}$, $TC_0(X) = Z(f^{\text{in}}) = Z(f)$, and $\mathbb{P}TC_0(X)$ is the smooth conic in $\mathbb{P}^2$.

There are a lot of instances where properties of our variety are dictated by their projectivized tangent cone. For example, a singularity of a variety is called **ordinary** if the projectivized tangent cone with respect to that singularity is smooth.

We have some corollaries of the homework exercise listed above.

Corollary 19.6

$\dim TC_x X = \dim_x X$.

Note that $\dim_x X = \dim X$ if X is irreducible by definition.

Proof. $\dim \widetilde{X} = \dim X = \dim E + 1 = \dim \mathbb{P}TC_x X + 1 = \dim TC_x X$. □

Corollary 19.7

Let X be irreducible. Then, $x \in X$ is a smooth point iff $TC_x X = T_x X$.

19.1 Intrinsic Interpretation

Let X be a variety and $x \in X$. We have the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$. We can take the $\mathfrak{m}_x$-adic filtration

$$\mathcal{O}_{X,x} \supseteq \mathfrak{m}_x \supseteq \mathfrak{m}_x^2 \supseteq \mathfrak{m}_x^3 \supseteq \dots$$

Remark 19.8. Because Popa asked this in class, I'll put it here: $\bigcap_{k \in \mathbb{N}} \mathfrak{m}_x^k = 0$.

We can associate this with the graded ring $R = \operatorname{gr}(\mathcal{O}_{X,x}) = \bigoplus_{i=0}^{\infty} \mathfrak{m}_x^i / \mathfrak{m}_x^{i+1}$. Denote $R_i = \mathfrak{m}_x^i / \mathfrak{m}_x^{i+1}$, so $R_0 = k$ and $R_1 = T_x^* X$. We see that R is generated by R_1 as a k-algebra.

Exercise 19.9. (Also homework) If $X \subseteq \mathbb{A}^n$ is affine (and $x = 0$), the mapping $k[X_1, \dots, X_n]/I(X)^{\text{in}} \to R$ sending $\overline{X_i} \to \widehat{X_i}$ ($\widehat{X_i}$ is the image of X_i in $\mathfrak{m}_x/\mathfrak{m}_x^2$) is a k-algebra isomorphism.

Because of this result, we can say $TC_x X$ has affine coordinate ring R.

Let's take a step back and appreciate these abstract constructions a little more. We start with the quotient $\mathfrak{m}_x/\mathfrak{m}_x^2$. Letting $k = \mathcal{O}_{X,x}/\mathfrak{m}_x$, we see that this is a k-vector space, isomorphic to the dual $(T_xX)^*$. We can form the symmetric algebra $\mathrm{Sym}^k(\mathfrak{m}_x/\mathfrak{m}_x^2)$ (recall $\mathrm{Sym}^k V = V^{\otimes k}/(x \otimes y - y \otimes x)$). In general, $\mathrm{Sym}^\bullet V = \bigoplus_{i\geq 0} \mathrm{Sym}^k V$ is a polynomial ring; explicitly, if we fix a basis $x_1, \dots, x_n$ of V, then $\mathrm{Sym}^\bullet V \simeq k[x_1, \dots, x_n]$. (Explanation: the basis elements have no relation with each other except that they commute, which is exactly the polynomial ring.)

But we can easily realize $k[x_1, \dots, x_n]$ as the set of functions on $\mathbb{A}^n$. Considering $\mathrm{Sym}^\bullet V^*$, we see that it is the algebra generated by linear functionals, aka it is the set of functions on V. We will apply this to our tangent space situation.

One can find a natural projection map $\mathrm{Sym}^k(\mathfrak{m}_x/\mathfrak{m}_x^2) \twoheadrightarrow \mathfrak{m}_x^k/\mathfrak{m}_x^{k+1}$. (Check this! Exercise.) This gives us a surjection

$$A = \mathrm{Sym}^\bullet(\mathfrak{m}_x/\mathfrak{m}_x^2) = \bigoplus_{i\geq 0} \mathrm{Sym}^i(\mathfrak{m}_x/\mathfrak{m}_x^2) \twoheadrightarrow \bigoplus_{i\geq 0} \mathfrak{m}_x^i/\mathfrak{m}_x^{i+1} = R.$$

The thing on the right is, by the homework exercise, the affine coordinate ring of the tangent cone. The left is the coordinate algebra of $T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. This is thus a surjection on the coordinate rings, which pulls back to an inclusion of projective varieties $TC_xX \subseteq T_xX$. This is quite powerful because we *never* mention here any embedding of X into an ambient (affine) space; we only rely on the germ of functions, from which we can proceed with purely algebraic methods.

20 04/19 - Hilbert Polynomials

Recorded lecture since Popa was away; I will fill in this later.

21 04/24 - More on Hilbert Polynomials

Here we recall some important definitions. Let $X \subseteq \mathbb{P}^n$ be an algebraic set. Then, $I(X)$ is a homogeneous radical ideal in $k[x_0, \dots, x_n]$. This gives us a graded ring

$$S(X) = k[x_0, \dots, x_n]/I(X) = \bigoplus_{d\geq 0} S(X)_d,$$

where each degree-d part $S(X)_d$ is a finite dimensional vector space over k. (Sanity check: verify it's finite dimensional by finding a spanning list.)

We use this to define a Hilbert function.

> **Definition 21.1** (Hilbert Function). The **Hilbert function** of X (in $\mathbb{P}^n$) is a map $h_X : \mathbb{N} \to \mathbb{N}$ such that $h_X(d) = \dim_k S(X)_d$.

Remark 21.2. We can define a Hilbert polynomial for any homogeneous (not necessarily radical) ideal $I \subseteq k[x_0, \dots, x_n]$. In this case, we can identify $I = I(X)$ for some *projective scheme* X. In this setting, we have $I(X \cap Y) = I(X) + I(Y)$. This setup works even more generally for any finitely generated graded module M over $S := k[x_0, \dots, x_n]$.

Example 21.3 (Hilbert functions)

Start simple. Let $X = \mathbb{P}^n$, then $S(X) = k[x_0, \dots, x_n]$, where $S(X)_d$ are the homogeneous polynomials of degree d. We have $h_X(d) = \binom{n+d}{n}$, which we can treat as a polynomial in d of degree n with leading coefficient $1/n!$.

Consider another simple case in the opposite end of the spectrum. Let $X = \{(1 : 0), (0 : 1)\} \subseteq \mathbb{P}^1$. Then, $I(X) = (X_0X_1)$, so the basis of $S(X)_d$ is either just 1 for $d = 0$ or $\{X_0^d, X_1^d\}$ for $d \geq 1$. This gives the function $h_X(d) = 1$ when $d = 0$ or 2 otherwise.

Because we are expanding our attention to include schemes (i.e. when $I(X)$ is not radical), let's consider the "double point" scheme $X \subseteq \mathbb{P}^1$. (As a set, you're kinda counting the point $(0 : 1)$ twice, but it's easier to talk about its vanishing ideal.) We have $I(X) = (X_0^2)$, so the basis for $S(X)_d$ is 1 if $d = 0$ and $\{X_0, X_1^{d-1}, X_1^d\}$ otherwise. This means the Hilbert function is the same as above.

Finally, let $X = \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\} \subseteq \mathbb{P}^2$. (We could more generally take any three non-collinear points.) Then, $I(X) = (X_0X_1, X_0X_2, X_1X_2)$. We can construct a basis for $S(X)_d$: 1 for $d = 0$, and $\{X_0^d, X_1^d, X_2^d\}$ otherwise. Then, $h_X(d) = 1$ for $d = 0$ and 3 otherwise.

Exercise 21.4. (Going off of the last example above) Show that if X is a set of three collinear points in $\mathbb{P}^2$, then

$$h_X(d) = \begin{cases} 1 & d = 0 \\ 2 & d = 1 \\ 3 & d \geq 2 \end{cases}.$$

We see a pattern here. For all of these cases where X is finite, the Hilbert function stabilizes quickly to a value which coincides with the number of points in our scheme. We now formalize this:

Lemma 21.5

Let $X \subseteq \mathbb{P}^n$ be a 0-dimensional subset ("subscheme"). Then,

1. X is an affine algebraic set.
2. Let $R := A(X)$ be the affine coordinate ring. (Affine from above.) R is a finite dimensional vector space over k. (We call $\dim_k R$ as the **length of X**.)
3. $h_X(d) = \dim_k R$ for $d >> 0$.

Proof. (1) We can always find a hyperplane $H \subset \mathbb{P}^n$ such that $H \cap X - \emptyset$. (The set of all hyperplanes passing through a point is closed in the set of all hyperplanes, so the set of all planes passing through any point in X is a finite union of closed subsets, so it is closed.) Then, $X \subseteq \mathbb{P}^n \setminus H = \mathbb{A}^n$, woohoo.

(2) This was a homework exercise,[2] so we move on, hahahaha.

(3) We may assume that H in (1) is $H = (X_0 = 0)$. Let $\overline{f_1}, \dots, \overline{f_\ell}$ be a k-basis of R, with $\deg f_i = d_i$. Let $d \geq \max_j\{d_j\}$, and consider the k-linear maps

$$\begin{aligned} \alpha : S(X)_d &\to R \\ F &\mapsto \overline{f}(x_1, \dots, x_n) = F(1, x_1, \dots, x_n). \\ \beta : R &\to S(X)_d \\ \overline{f} &\mapsto \overline{f}^{\text{hom}} \cdot X_0^{d - \deg f}. \end{aligned}$$

It is straightforward to check that α and β are inverses of each other. This gives us an isomorphism $R \cong S(X)_d$ over k, and the conclusion follows. ☐

Now we present the main result on Hilbert polynomials, namely that they exist: any Hilbert function is actually a polynomial.

Theorem 21.6

Let $X \subseteq \mathbb{P}^n$ be an m-dimensional algebraic set (or projective subscheme). Then, there exists a unique polynomial $\chi_X \in \mathbb{Q}[T]$ such that $h_X(d) = \chi_X(d)$ for $d >> 0$. Moreover,

1. $\deg \chi_X = m$,
2. the leading coefficient of χ_X is $1/m! \cdot N$ for some positive itneger N.

Definition 21.7 (Hilbert Polynomial)**.** We call χ_X the **Hilbert polynomial** of $X \subseteq \mathbb{P}^n$.

Proof. Induct on m. This takes advantage of the fact that we proved this for $m = 0$ in the previous lemma; our base case is complete.

Choose a general hyperplane such that no component of X is contained in H. By some change of coordinates, we may assume $H = (X_0 = 0)$. we now have a short exact sequence of $k[X_0, \dots, X_n]$-modules:

$$0 \to k[X_0, \dots, X_n]/I(X) \xrightarrow{X_0} k[X_1, \dots, X_n]/I(X) \to k[X_0, \dots, X_n]/(I(X) + (X_0)) \to 0.$$

Note that $I(X) + (X_0) = I(X \cap H)$. The first map X_0 is injective, otherwise there exists homogeneous F such that $X_0 \circ F \in I(X)$, in which case $X \subseteq (X \cap Z(F)) \cup (X \cap H)$. But since $F \notin I(X)$, we get $X \subseteq X \cap Z(F)$, a contradiction.

[2] Problem 5 on Problem Set 3

This exact sequence is nice because now we can take dimensions and relate the to our Hilbert polynomials:

$$h_X(d) = h_X(d-1) + h_{X\cap H}(d).$$

By induction, $h_{X\cap H}(d)$ is a polynomial of degree $(m-1)$ (for $d >> 0$), with leading coefficient $\frac{1}{(m-1)!} \cdot N$ for some positive integer N.

We can write $h_{X\cap H}(d) = \sum_{i=0}^{m-1} c_i \binom{d}{i}$, where $c_i \in \mathbb{Q}$, $c_{m-1} \in \mathbb{N}^*$. (In fact, if we give a more full proof of this, then we can obtain $c_i \in \mathbb{Z}$ and not just in $\mathbb{Q}$, but we do not need this distinction in our proof. See Hartshorne Proposition 7.3 for a reference.) A very brief argument: obtain the c_i values by induction downwards. c_{m-1} is determined; we can determine c_{m-2} by looking at the degree-$(m-2)$ part, where the coefficient is just in terms of c_{m-1} and c_{m-2}.

Now, we present the final stamp.

Claim 21.8. For $d >> 0$,

$$h_X(d) = c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1}$$

for some constant c.

This finishes our proof, so we just need to prove the claim. As intuition for why this works, we have the nice identity $\binom{d+1}{i+1} = \binom{d}{i+1} + \binom{d}{i}$. We want to make use of the equation of Hilbert polynomials we obtained from our short exact sequence. Let $f(d) = h_X(d)$ and $P(d) = \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1}$.

Define $\Delta f(d) = f(d) - f(d-1)$, and likewise for $\Delta P(d)$. The equation from the short exact sequence tells us $\Delta h_X(d) = h_{X\cap H}(d)$. By the identity on binomial coefficients, we have $\Delta P(d) = \sum_{i=0}^{m-1} c_i \binom{d}{i}$, which is our expression for $h_{X\cap H}(d)$. Thus, $\Delta(f-P) = 0$ for $d >> 0$, which means $f - P = c$ constant, and we conclude. □

Definition 21.9 (Degree of algebraic set)**.** Let $X \subseteq \mathbb{P}^n$ be an algebraic set (subscheme). The **degree of X** is the product of $(\dim X)!$ and the leading coefficient of $\chi_X(d)$.

Example 21.10 (Degrees)

Since $\chi_{\mathbb{P}^n} = \frac{1}{n!} d^n + \dots$, we have $\deg(\mathbb{P}^n) = 1$. If $\dim X = 0$, then $\chi_X(d) = \text{length} \cdot d^0$, so $\deg(X) = \text{length}(X)$.

Another example, but we need longer justification for this.

Claim 21.11. If $X = Z(F) \subseteq \mathbb{P}^n$ is a hypersurface, then $\deg X = \deg F$.

Proof. We have a short exact sequence $0 \to S \xrightarrow{F} S \to S/(F) \to 0$. ($S := k[X_0, \ldots, X_n]$) This gives us

$$\begin{aligned} h_X(d) &= h_{\mathbb{P}^n}(d) - h_{\mathbb{P}^n}(d - \deg F) \\ &= \binom{d+n}{n} - \binom{d - \deg F + n}{n} \\ &= \frac{(d+n)\cdots(d+1)}{n!} - \frac{(d - \deg F + n)\cdots(d - \deg F + 1)}{n!} \\ &= \frac{\deg F}{(n-1)!} \cdot d^{n-1} + \text{lower order terms}, \end{aligned}$$

and the end is clear. □

But wait, we already define degree before! (Definition 15.12) Luckily, these two notions of degree coincide; this will be a final exam problem.

22 04/26 - Bezout's Theorem

Start with the setup we had from last time. Let $X \subseteq \mathbb{P}^n$. This gives a homogeneous $I(X) \subseteq k[X_0, \ldots, X_n]$, which in turn gives us a Hilbert function h_X, which is actually a polynomial χ_X of the form

$$\chi_X(d) = \frac{\deg X}{(\dim X)!} \cdot d^{\dim X} + \text{lower order terms}.$$

From here, the genus of a curve arises in a very natural way. For a projective curve $C \subseteq \mathbb{P}^n$, we have $\chi(\mathcal{O}_C(d)) = (\deg C) \cdot d - r$ for some constant r. We set $g = r + 1$ as the **genus** of C.

Definition 22.1. The **arithmetic genus** of any projective variety $X \subseteq \mathbb{P}^n$ is

$$p_a(X) = (-1)^{\dim X}(\chi_X(0) - 1).$$

Remark 22.2. This is independent of the embedding in $\mathbb{P}^n$, so this is an invariant intrinsic to the variety, which is a powerful thing.

Proposition 22.3

Let $X, Y \subseteq \mathbb{P}^n$ be varieties of dimension m such that $\dim X \cap Y < m$. Then, $\deg X \cup Y = \deg X + \deg Y$.

An example of this scenario to keep in mind is if two irreducible components intersect, then their intersection will have smaller intersection, so you can just add degrees freely without worrying about the intersection. For instance, two curves may intersect at some points, but there are just finitely many, so its dimension is 0 and we can add degrees.

Proof. Recall $I(X \cap Y) = I(X) + I(Y)$ and $I(X \cup Y) = I(X) \cap I(Y)$. In general, for any ideals $I_1, I_2 \subset R$, we always have a short exact sequence

$$0 \to R/I_1 \cap I_2 \to R/I_1 \oplus R/I_2 \to R/I_1 + I_2 \to 0,$$

where the first nontrivial map is $\overline{f} \mapsto (\overline{f}, \overline{f})$ and the second is $(\overline{f}, \overline{g}) \mapsto \overline{f} - \overline{g}$. This gives us the short exact sequence

$$0 \to S/I(X) \cap I(Y) \to S/I(X) \oplus S/I(Y) \to S/I(X) + I(Y) \to 0,$$

where $S = k[X_0, \dots, X_n]$. Taking dimensions, we have

$$h_X(d) + h_Y(d) = h_{X \cup Y}(d) + h_{X \cap Y}(d).$$

Looking at the leading terms (in particular, the coefficient of d^m), we get

$$\frac{\deg X}{m!} + \frac{\deg Y}{m!} = \frac{\deg(X \cup Y)}{m!} + 0,$$

which gives us the desired result. □

Note that this is the way we want to proceed with a lot of these Hilbert polynomial arguments: find a short exact sequence, then take dimensions to actually get a statement about Hilbert polynomials.

22.1 The Proof

Now we're ready to tackle (a slightly weaker, but still powerful version of) the theorem we've been holding on to for a really long time: Bezout's Theorem.

Theorem 22.4 (Bezout's Theorem)
If $X \subseteq \mathbb{P}^n$, $\dim X > 0$, and $F \in k[X_0, \dots, X_n]$ is a homogeneous polynomial such that no component of X is contained in $Z(F)$, then

$$\deg(X \cap Z(F)) = \deg X \cdot \deg F.$$

If you look at Hartshorne Theorem 7.7, you see there's one more aspect to this statement. We can generalize the notion of intersection multiplicity to higher dimensions (the definition of this becomes something purely commutative algebra). Then, $\deg(X \cap Z(F))$ is the sum of the intersection multiplicities. This is the grown-up version of Theorem 3.8. We will exclude this discussion, though, since it involves a more lengthy commutative algebra exposition that we just don't have time for.

Proof. Let $S := k[X_0, \dots, X_n]$. We have a short exact sequence

$$0 \to S/I(X) \xrightarrow{F} S/I(X) \to S/(I(X) + (F)) \to 0,$$

which yields the equation

$$\chi_{X\cap Z(F)}(d) = \chi_X(d) - \chi_X(d - \deg F).$$

Expanding, the right hand side looks like

$$\begin{aligned}\chi_X(d) - \chi_X(d-\deg F) &= \frac{\deg X}{m!}(d^m - (d-\deg F)^m) \\ &\quad + c_{m-1}(d^{m-1} - (d-\deg F)^{m-1}) + \text{l.o.t} \\ &= \frac{\deg X}{(m-1)!}\cdot \deg F \cdot d^{m-1} + \text{l.o.t.},\end{aligned}$$

and so comparing the coefficients of the d^{m-1} term gives the desired result. □

Example 22.5

This gives us Theorem 1.2: if $C_1, C_2 \subseteq \mathbb{P}^2$ with no common components, then $\deg(C_1 \cap C_2) = \deg C_1 \cdot \deg C_2$.

22.2 Applications of Bezout's Theorem

Let's look at just how powerful this statement is by going through some of its consequences. Some may seem surprising or not related to Bezout's at all!

Corollary 22.6

Every isomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ is linear, i.e. $f(\underline{x}) = A \cdot \underline{x}$ for some $A \in \mathrm{GL}_{n+1}(k)$.

Proof. Let $H, L \subset \mathbb{P}^n$ such that L is a line and $L \not\subseteq H$. Then, $H \cap L$ is just a point. As f is an isomorphism, $f(H) \cap f(L)$ is also just a point. By Bezout's Theorem, $1 = \deg(f(H) \cap f(L)) = \deg f(H) \cdot \deg f(L)$, so $\deg f(H) = \deg f(L) = 1$. In particular, $\deg f(H) = 1$ means H is a hyperplane. Applying this to $H_i = (X_i = 0)$, we have $X_i \mapsto \sum_{j=1}^n a_{ij} X_i$ is an isomorphism, so $A = (a_{ij})_{i,j}$ is invertible. □

Bezout's Theorem is often used to count "bad things." Here's a good example:

Corollary 22.7

Let $C \subseteq \mathbb{P}^2$ be an irreducible curve of degree d. Then, there exists at most $\binom{d-1}{2}$ singular points on C.

Thus, any cubic can have at most one singular point.

Proof. Assume the contrary. Then, there exists $P_1, \dots, P_{\binom{d-1}{2}+1}$ singular points in C. Choose $Q_1, \dots, Q_{d-3}$ other distinct points on C. In total, then, we have $\binom{d-1}{2} + 1 + d - 3 = \binom{d}{2} - 1$ points.

Claim 22.8. There exists a curve C' of degree $(d-2)$ passing through all the P_i's and Q_j's.

Proof. Note that the set of hypersurfaces of degree m in $\mathbb{P}^n$ can be identified with $\mathbb{P}^{\binom{n+m}{m}-1}$. Thus, the set of curves of degree $d-2$ in $\mathbb{P}^2$ can be identified with $\mathbb{P}^{\binom{d}{2}-1}$. The set of all curves of degree $d-2$ in $\mathbb{P}^2$ passing through a specific point (say, P_1) is a hyperplane H_{P_1}, as this is just solving for the coefficients in $\sum a_I \cdot \underline{X}_0^I = 0$. But given $\binom{d}{2} - 1$ hyperplanes in $\mathbb{P}^{\binom{d}{2}-1}$, their intersection is non-empty, which means there exists a curve passing through all P_i's and Q_j's, as desired. □

Now we can apply Bezout's Theorem to C and C'. We can look at $\deg(C \cap C')$ in two different way. First, we apply Bezout's, from which we get $\deg C \cdot \deg C' = d(d-2)$. The other is via the sum of the intersection multiplicities (which we technically did not prove, but trust that it is true – it's in Hartshorne, Theorem 7.7), from which we get

$$\begin{aligned}\deg(C \cap C') &\geq (d-3) \cdot 1 + \left(\binom{d-1}{2} + 1\right) \cdot 2 \\ &= d - 3 + 2 + (d-1)(d-2) \\ &= d^2 - 2d + 1,\end{aligned}$$

where the $(d-3)$ comes from the Q_j's and the other term comes from the P_i's. But this is impossible, unless C and C' have a common component. By irreducibility of C, this means $C \subseteq C'$, but $\deg C' < \deg C$, a contradiction. □

Well, that was a really good run everybody! There's just so much in algebraic geometry, but hopefully this class has (1) provided you with necessary background to keep working on these things, and (2) given you a glimpse of how cool the subject can be. Good luck on the exam!