Representation Theory

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1 Preface

In brief, representation theory approaches abstract algebra via linear algebra. This is useful because while we do have an extensive understanding of abstract algebra, we understand linear algebra the best. As Toyesh says, "all of math is attempting to translate things into linear algebra."

One might think, though, that this reduction from abstract algebra to linear algebra is not fruitful: as much as we are more comfortable with linear algebra, this translation may lose lots of information. The beautiful crux of representation theory is that, actually, lots of information about the group (resp., ring, module) can be recovered from its representations. This is not an evident thing at all, but this result is the basis of the Tannakian philosophy. Tannaka's Theorem, for instance, allows us to recover a compact Lie group by its category of representations.

For this handout, we will focus on the representations of finite groups. The assumption of finiteness is quite a strong one and allows us to do a lot of things which we otherwise cannot afford for infinite groups. One may think this finiteness condition is too restrictive, but the arguments for representations of finite groups translate well to those of compact Lie groups: in short, change all summations into integrals, which is possible by the existence of a Haar measure from compactness.

The study of representations of infinite groups in general is an ongoing topic of research. The representations of Lie groups is rich; we can understand them well through the study of Lie algebras and their representations.

2 The Fundamentals

We now define a representation.

Definition 2.1 (Representation)**.** Let G be a finite group. A (complex) **representation** of G is a pair (V, ρ), where V is a vector space over $\mathbb{C}$ and ρ is a group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

Here, $\mathrm{GL}(V)$ is the set of automorphisms of V, i.e., invertible linear operators from V to itself.

Remark 2.2. For this handout, we will assume all representations are complex representations, although it makes sense to talk about real representations or, more generally, representations over any field. For instance, Emma Knight's guest counselor seminar will be about representations of p-groups over $\mathbb{F}_q$ for some prime power $q = p^k$.

Definition 2.3. The **dimension** of a representation (V, ρ) is simply $\dim_{\mathbb{C}} V$.

Remark 2.4. We additionally study only finite-dimensional representations in this handout. Infinite-dimensional representations can be handled, and in fact they decompose (whatever that means) in the same way as finite-dimensional representations do for finite groups, but they are just more annoying to deal with.

Perhaps a more enlightening way of thinking about representations is considering them as a group action on a vector space. Consider some $g \in G$. The map given by the action of g is itself a map $g\cdot : V \to V$ where $v \mapsto g \cdot v$. Furthermore, this map has an inverse, given by the action of g^{-1} since $g^{-1} \cdot (gv) = g \cdot (g^{-1}v) = v$. This allows us to interpret the group action $G \times V \to V$ as the group homomorphism $G \to \mathrm{GL}(V)$ where, as described, g gets sent to the map $v \mapsto g \cdot v$.

Remark 2.5. There are many ways to notate a representation (V, ρ):

- Oftentimes, we will only refer to a representation by its vector space V or its homomorphism ρ. Other times, V_ρ serves as shorthand notation, where the vector space V_ρ implicitly comes with a homomorphism ρ. In any case, it should be understood that a representation contains both the data of a vector space *and* a group homomorphism, even if only one of them is mentioned.
- The automorphism $\rho(g)$ is often denoted as ρ_g.
- We can also notate a representation by interpreting it as a group action. Thus, as the notation $\rho(g)(v)$ is quite clunky, we can write the image of v under $\rho(g)$ as simply $g \cdot v$, or just gv.

When I first learned about representations, they felt very non-intuitive to me. For instance, I was only familiar with group actions on a set, usually finite, so it was difficult to imagine a group action on a vector space. Luckily, a (finite-dimensional) vector space comes with a (finite) set: its basis. The first two examples of representations will be given by the action on its bases.

Example 2.6 (Permutation Representation)**.** Consider the symmetric group S_3. Let V be a 3-dimensional vector space with basis $\{e_1, e_2, e_3\}$. Then, we have a representation

$$\begin{aligned} \rho : S_3 &\to \mathrm{GL}(V) \\ \sigma &\mapsto (e_i \mapsto e_{\sigma(i)}). \end{aligned}$$

For instance, $\rho_{(12)}(e_1 + e_3) = e_2 + e_3$. We can similarly define an n-dimensional representation of S_n.

Even more generally, given an action of G on a finite set $X = \{x_1, x_2, \dots, x_n\}$, we can construct an n-dimensional representation V of G which has basis $\{e_{x_1}, \dots, e_{x_n}\}$. The basis elements permute based on the group action, namely, $g \cdot e_{x_i} = x_{g \cdot x_i}$. For instance, the group $\mathbb{Z}/3\mathbb{Z}$ acts on the basis $\{e_1, e_2, e_3\}$ by $a \cdot e_i = e_{i+a}$, where the indices are taken modulo 3.

Example 2.7 (Regular Representation)**.** This is a specific case of the permutation representation when our finite set X is the set G itself, and the action is just left multiplication. Let $G = \{g_1, \dots, g_n\}$. Then, we can construct an n-dimensional representation of G which has basis $\{e_{g_1}, \dots, e_{g_n}\}$ and is given by $h \cdot e_{g_i} = e_{hg_i}$.

Example 2.8 (Trivial Representation)**.** Every group G admits a **trivial representation**, where each group element gets sent to the identity map. Technically we can have this for any vector space V, but we usually refer to the trivial representation specifically for the dimension-1 case $V = \mathbb{C}$. The reasons for this will become clear once we talk about irreducible representations.

Example 2.9 (Alternating Representation)**.** For the symmetric group S_n, we have the one-dimensional representation $\mathrm{alt} : S_n \to \mathrm{GL}(\mathbb{C}) \simeq \mathbb{C}^\times$ where $\mathrm{alt}(\sigma) = \mathrm{sgn}(\sigma)$, the sign of the permutation σ.

Example 2.10 (Adjoint Representation)**.** Let $G = \mathrm{GL}_n(\mathbb{C})$. We can consider the vector space V consisting of all $n \times n$ matrices with complex coefficients. For any matrix $g \in G$, define the functional $\Psi_g : V \to V$ mapping $X \mapsto gXg^{-1}$. Then, we have the **adjoint representation**

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{GL}(V) \\ g &\mapsto \Psi_g. \end{aligned}$$

This is a *very important tool in the representation theory of Lie groups*! More generally, given a Lie group G, we can take the corresponding Lie algebra $\mathfrak{g} = T_e G$, the tangent space of G at the identity. This is a vector space, and for every $g \in G$, we have a functional $\Psi_g : T_e G \to T_e G$ mapping $X \mapsto gXg^{-1}$ as in the $G = \mathrm{GL}_n(\mathbb{C})$ case. Then, we can define the adjoint representation as above. This representation is important because the differential of Ad at the identity is the adjoint representation of the *Lie algebra* $\mathfrak{g} := T_e G$, which we use to define the Lie bracket on $\mathfrak{g}$.

Given two representations, there are two nifty ways to construct a new representation: the direct sum and the tensor product. We know they exist for vector spaces, but we are able to define them to still observe the representation structure.

Definition 2.11 (Direct Sum of Representations)**.** Given representations V and W of G, we can realize $V \oplus W$ as a representation of G via $g \cdot (v \oplus w) = gv \oplus gw$.

It helps to think in terms of matrices. If (V, ρ) and (W, σ) are our representations, and we consider $\rho_g \in \mathrm{GL}(V)$ and $\sigma_g \in \mathrm{GL}(W)$ as matrices, then we can associate $(\rho \oplus \sigma)_g$ with the matrix

$$\begin{bmatrix} \rho_g & 0 \\ 0 & \sigma_g \end{bmatrix},$$

where each entry is a block matrix.

Definition 2.12 (Tensor Product of Representations)**.** Letting V, W be representations of G again, we define the representation $V \otimes W$ via $g \cdot (v \otimes w) = gv \otimes gw$.

We will also see this tensor product representation through matrices. Consider the representations (V, ρ) and (W, σ), with dimensions n and m respectively. Suppose $\rho_g = (r_{ij}(g))_{i,j}$ (after we fix some basis for V) and $\sigma_g = (s_{ij}(g))_{i,j}$ (after we fix some basis for W). Then, $(\rho \otimes \sigma)_g$ can be represented by the matrix whose entries are products $r_{i_1 j_1}(g) s_{i_2 j_2}(g)$. For instance, if $\rho_g = \left[\begin{smallmatrix} 3 & 2 \\ 1 & 1 \end{smallmatrix}\right]$ (with respect to some basis $\{e_1, e_2\}$ of V) and $\sigma_g = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 6 & 0 & 1 \end{smallmatrix}\right]$ (with respect to some basis $\{f_1, f_2, f_3\}$ of W), then we have

$$\begin{aligned} (\rho \otimes \sigma)_g (e_1 \otimes f_2) &= \rho_g(e_1) \otimes \sigma_g(f_2) \\ &= (3e_1 + 2e_2) \otimes (4f_1 + 5f_2 + f_3) \\ &= 12e_1 \otimes f_1 + 15e_1 \otimes f_2 + 3e_1 \otimes f_3 + \\ &\qquad 8e_2 \otimes f_1 + 10e_2 \otimes f_2 + 2e_2 \otimes f_3, \end{aligned}$$

which agrees with our general matrix form for the action of g on the tensor product.

We can also take the dual of a representation to produce another representation, which we now construct.

Example 2.13 (Contragredient/Dual Representation)**.** Let (V, ρ) be a representation of G. The **contragedient** (or less pretentiously, the **dual**) representation of V is the representation $(V^\vee, \rho^\vee)$ such that, for $\phi \in V^\vee$,

$$\rho_g^\vee(\phi)(\rho_g(v)) = \phi(v).$$

Equivalently, by replacing v with $\rho_g^{-1}(v)$, we have the definition $\rho_g^\vee(\phi) = \phi \circ \rho_{g^{-1}}$.

We will now indulge in the two instinctive moves when studying any new mathematical object: 1) study maps between the objects, and 2) find the "simplest" kind of these objects.

We first study maps between representations. Let (V, ρ) and (W, σ) be two representations of G. Then, a map between these representations is a linear map $f : V \to W$ which is G-equivariant. Explicitly, this means $f : V \to W$ satisfies the following commutative diagram for every $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[\sigma_g]{} & W \end{array}$$

The map f is an **isomorphism** if it is invertible, in which case V and W are isomorphic as representations. Many of the isomorphisms we know to be true on the level of vector spaces remain isomorphisms as representations.

Example 2.14 (Hom as Tensor)**.** Let (V, ρ) and (W, σ) be representations of G. As vector spaces, we know $\text{Hom}(V, W) \simeq V^\vee \otimes W$. Given our definitions thus far, we have an understanding of $V^\vee \otimes W$ as a representation. For the left hand side, we define the group action of G on $\text{Hom}(V, W)$ by

$$g \cdot \phi = \sigma_g \circ \phi \circ \rho_g^{-1}.$$

Note that by definition, G acts trivially on the G-equivariant maps in $\text{Hom}(V, W)$. (In the above commutative diagram, the f on the left side is mapped by the g-action to the f on the right.)

We now claim that the vector space isomorphism $V^\vee \otimes W \simeq \text{Hom}(V, W)$ given by $\phi \otimes w \mapsto (v \mapsto \phi(v)w)$ is also an isomorphism of representations. This amounts to checking that the diagram

$$\begin{array}{ccc} V^\vee \otimes W & \xrightarrow{\rho_g^\vee \otimes \sigma} & V^\vee \otimes W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \text{Hom}(V, W) & \xrightarrow[g \cdot -]{} & \text{Hom}(V, W) \end{array}$$

commutes, which we can do manually. Given some $\phi \otimes w \in V^\vee \otimes W$, we have

$$\begin{aligned} f(g \cdot (\phi \otimes w))(v) &= f(\phi \circ \rho_g^{-1} \otimes gw)(v) \\ &= \phi \circ \rho_g^{-1}(v) gw \\ g \cdot f(\phi \otimes w)(v) &= \sigma_g \circ f(\phi \otimes w) \circ \rho_g^{-1}(v) \\ &= \sigma_g(\phi \circ \rho_g^{-1}(v) w) \\ &= \phi \circ \rho_g^{-1}(v) gw, \end{aligned}$$

so indeed the diagram commutes.

3 Irreducible Representations

Our second mathematical instinct is to obtain some notion of irreducibility. How can we define an irreducible representation, and can we "decompose" any representation into irreducibles? One very simple example we can see is the case of the trivial representation. For any vector space V, we can define $\rho : G \to \text{GL}(V)$ to send everything to the identity id_V. But we could really just see any one-dimensional space of V to see this group action in play.

The first step to defining an irreducible representation is defining a subrepresentation, which will allow us to begin decomposing representations.

Definition 3.1 (Subrepresentation)**.** Let (V, ρ) be a representation. A **subrepresentation** W is a subspace of V which is invariant under the action by G, i.e., every ρ_g restricts to an automorphism of $W \subseteq V$.

Example 3.2 (Trivial Representation as Subrepresentation)**.** Let $G \to \mathrm{GL}(V)$ map every element to id_V. Then, any one-dimensional subspace of V will be *the* trivial representation, which we can see as a subrepresentation of V. Another example of the trivial representation arising as a subrepresentation is for the permutation representation $S_n \to \mathrm{GL}_n(\mathbb{C})$. (Here, the vector space is implicitly $\mathbb{C}^n$ with basis $\{e_1, \dots, e_n\}$.) The vector $e_1 + e_2 + \cdots + e_n$ is invariant under the whole S_n-action, so $\mathrm{span}(e_1 + \cdots + e_n)$ is a trivial subrepresentation.

Example 3.3 (Dimension 2 Subrepresentation of S_3)**.** Consider the permutation representation of S_3, namely the representation $\mathbb{C}^3$ which permutes the basis elements $\{e_1, e_2, e_3\}$. Then, one can check that the subspace $W = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$ is invariant under the S_3-action. Verify for yourself that this subrepresentation has dimension 2. This is known as the **standard representation** of S_3, which we will revisit when classifying all irreducible representations of S_3.

The notion of irreducibility for representations can now be expressed in the obvious way.

Definition 3.4 (Irreducible Representation)**.** A representation V of G is an **irreducible representation** if there are no proper subrepresentations of V.

Exercise 3.5. Check that the trivial representation is irreducible. Check that the standard representation of S_3, the dimension-2 representation from Example 3.3, is irreducible as well.

Our ambitions would be fulfilled if we could express every representation as a direct sum of irreducible representations. Equivalently, for any subrepresentation $W \subset V$, we wish to find a complement $W^\perp$ of W such that $W \oplus W^\perp = V$ and $W^\perp$ is also G-invariant, i.e., it is also a subrepresentation.

Exercise 3.6 (Summing Irreducibles)**.** Let $V_0 = \mathrm{span}(e_1 + e_2 + e_3)$ be the trivial subrepresentation in the permutation representation of S_3, and let $W = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$ be the standard representation of S_3. Convince yourself that the permutation representation V is isomorphic to the direct sum representation $V_0 \oplus W$.

We must be careful about our choice of complement, though, as it is not true that any complement of a subrepresentation is also a subrepresentation. For instance, consider the permutation representation $V = \mathbb{C}^3$ of S_3 again, and let $V_0 = \mathrm{span}(e_1 + e_2 + e_3)$ be the trivial subrepresentation. The subspace $W' = \mathrm{span}\{e_1, e_2\}$ is a complement of V_0, as $\{e_1 + e_2 + e_3, e_1, e_2\}$ is a basis of $\mathbb{C}^3$, but it is clearly not invariant under S_3. On the other hand, the standard representation is both a complement of V_0 and S_3-invariant. So the choice of complement matters.

Despite this cautionary message, we are always able to find a complementary subrepresentation.

Theorem 3.7. Let V be a representation of G, and let $W \subset V$ be a subrepresentation. Then, there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$.

Proof. We will employ a trick that is common when proving results for representation theory: take something that is not necessarily G-invariant at first, then force it to be G-invariant, usually by some averaging technique. (In fact, we've already seen this in disguise: for the permutation representation of S_3, each subspace (e_i) is not S_3-invariant, but $(e_1 + e_2 + e_3)$ is a subrepresentation.)

Take some projection $\pi : V \to W$. Its kernel is some complement of W, and in fact, complements of W are in bijection with projections $V \twoheadrightarrow W$. Seeing π as an element of $\mathrm{Hom}(V, V)$, we can now take the average

$$\pi' = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi = \frac{1}{|G|} \sum_{g \in G} \rho_g \circ \pi \circ \rho_g^{-1}.$$

Note that both ρ_g and $\rho_g^{-1} = \rho_{g^{-1}}$ preserve W by definition of subrepresentation, so π' remains a projection onto W and therefore corresponds to a complement of W.

I claim that $W' := \ker \pi'$ is invariant under G, which would complete the proof. We only need to show the implication $\pi'(v) = 0 \implies \pi'(g \cdot v) = 0$. This follows from the fact that π' is G-equivariant, since for any $s \in G$,

$$\begin{aligned} s \cdot \pi' &= \rho_s \circ \pi' \circ \rho_s^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_s \rho_g \circ \pi' \circ \rho_{g^{-1}} \rho_{s^{-1}} \\ &= \frac{1}{|G|} \sum_{g \in G} \rho_{sg} \circ \pi' \circ \rho_{(sg)^{-1}} \\ &= \pi'. \end{aligned}$$

In particular, this yields $\rho_s \circ \pi' = \pi \circ \rho_s$, from which we get $\pi'(v) = 0 \implies \pi'(g \cdot v) = \rho_g \circ \pi'(v) = \rho_g(0) = 0$, as desired. ◻

This proof is nice because not only does it tell us that there exists an invariant complement, but it also instructs us on how to construct it by taking the average of the elements in the G-orbit of your original projection.

Corollary 3.8 (Existence of Decomposition into Irreducibles)**.** *Every representation V can be written as a direct sum of irreducible representations.*

Although we are guaranteed the existence of such a decomposition, and we have a mechanism of computing an invariant complement of an (irreducible) subrepresentation, the problem of finding an irreducible subrepresentation in the first place persists. Even worse, we have yet to show that this decomposition is even unique.

We can resolve all of our problems through **Schur's Lemma**, which I personally consider as the most fundamental result in representation theory. Very colloquially, Schur's Lemma states that *irreducible representations are incompatible with one another*. And just like how water and oil can be easily separated because of their chemical incompatibility, Schur's Lemma gives us hope that we have a feasible way to isolate irreducible representations.

Mathematically, Schur's Lemma tells us that the only non-zero maps existing between irreducible representations are isomorphisms given by multiplicatino-by-scalar maps. (We often call such a map as a **homothety**.)

Theorem 3.9 (Schur's Lemma)**.** Let (V, ρ) and (W, σ) be two irreducible representations of G. Let $f \in \operatorname{Hom}(V, W)$ be a G-equivariant map. Then,

1. If $(V, \rho) \not\simeq (W, \sigma)$, then $f \equiv 0$.
2. If $(V, \rho) \simeq (W, \sigma)$, then $f = \lambda \cdot \mathrm{id}_V$ for some scalar $\lambda \in \mathbb{C}$.

Proof. The main idea is that the kernel and image of an operator $f : V \to W$ can be seen not only as subspaces of V and W, respectively, but also as subrepresentations.

We first prove that $\ker f$ is a subrepresentation of V. For any $g \in G$, we have $f \circ \rho_g = \sigma_g \circ f$. Thus, if $f(v) = 0$, then $f(\rho_g(v)) = \sigma_g(f(v)) = \sigma_g(0) = 0$, implying $\rho_g(v) \in \ker f$ as well. Similarly, if $w = f(v)$ for some $v \in V$, then $\sigma_g(w) = \sigma_g(f(v)) = f(\rho_g(v))$, from which we deduce $\operatorname{Im} f$ is a subrepresentation of W.

By irreducibility of V, we have either $\ker f = 0$ or $\ker f = V$. The latter gives $f \equiv 0$. Irreducibility of W gives $\operatorname{Im} f = 0$ or $\operatorname{Im} f = W$. Again, the former gives $f \equiv 0$. Otherwise, we combine $\ker f = 0$ and $\operatorname{Im} f = W$ to get that f is an isomorphism between V and W.

To prove the last part of (2), we rely on the fact that our representations are over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we can find some eigenvalue λ of $f \in \mathrm{GL}(V)$. But then we have that $\ker(f - \lambda \cdot \mathrm{id}_V)$ is a nonzero subrepresentation of V, which forces $f = \lambda \cdot \mathrm{id}_V$ on all of V, as desired. □

Exercise 3.10. The last paragraph omits the check that the map $f - \lambda \cdot \mathrm{id}_V$ remains G-equivariant. If you understand the notion of equivariance, this fact should be easily obtainable.

I hyped up Schur's Lemma as the most fundamental result in representation theory, so I need to follow up by demonstrating its benefits.

Theorem 3.11 (Uniqueness of Decomposition into Irreducibles)**.** Any representation can be written uniquely as a direct sum of irreducible representations, up to ordering.

Proof. We follow the spirit of the proof of unique prime factorization from the number theory psets. Suppose V exhibits two distinct decompositions

$$V \cong \bigoplus_i V_i^{\oplus a_i} \cong \bigoplus_j W_j^{\oplus b_j}.$$

Denote f as the isomorphism between the two decompositions. For any irreducible representation $V_i \subset V$, the image $f(V_i)$ is not only a subrepresentation, but an irreducible one. (Justification: $\ker f|_{V_i} = 0$ since f is nonzero and V_i is irreducible, so $f(V_i) \simeq V_i$.) Suppose $f(V_i) = W_j$ for some j. Schur's Lemma then dictates that $V_i \simeq W_j$. We can then conclude by continuing to compare irreducible representations or via some inductive hypothesis, but in any case, the result follows. $\square$

We have now effectively reduced the study of representations to the study of irreducible representations. Our goal is to **classify all irreducible representations of some finite group** G. Different choices for G gives rise to interesting theory of its own – the counselor seminars alone will cover the cases when G is a symmetric group, a p-group over a field of characteristic p, and $\mathrm{GL}_n(\mathbb{F}_p)$.

Schur's Lemma is powerful enough for us to complete the study if irreducible representations for abelian groups. In this case, the representations are trivial:

Theorem 3.12 (Irreducible Reps of Abelian are Homotheties)**.** Any irreducible representation of an abelian group G has dimension one.

Proof. Let (V, ρ) be an irreducible representation of an abelian G. The crux here is that the commutativity of G forces each ρ_g to be a homothety, since $\rho_g \rho_h = \rho_{gh} = \rho_{hg} = \rho_h \rho_g$. But any homothety preserves any one-dimensional subspace, so the irreducibility of V forces it to be one-dimensional, as desired. $\square$

We have now provided enough setup to introduce a nice motivation for why we care about representation theory.

The study of Fourier series really comes from representation theory in disguise. Consider the representation $L^2(S^1)$ of S^1, where the group action is given by $\alpha \cdot f(z) = f(\alpha z)$. (Alternately, identifying $S^1 \simeq \mathbb{R}/\mathbb{Z}$, we are considering the representation on $\mathbb{R}/\mathbb{Z}$ of singly-periodic integrable functions.) Since S^1 is an abelian group, the irreducible representations are one-dimensional, namely they are (continuous) characters on $S^1 \simeq \mathbb{R}/\mathbb{Z}$. One can show using functional analysis that the only such characters are $\widehat{\mathbb{R}/\mathbb{Z}} \simeq \mathbb{Z}$ given by $\phi(a) = e^{2\pi i n a}$ for some $n \in \mathbb{Z}$. A Fourier series is exactly this: it expresses a periodic function as the sum of exponentials, or equivalently of trigonometric functions (which can be expressed as a sum of exponentials).

In a similar vein, considering the representation $L^2(\mathbb{R})$ of $\mathbb{R}$ gives us the study of Fourier transform. Attempting to do harmonic analysis on matrix groups like $\mathrm{GL}_n(\mathbb{R})$ leads to the study of reductive groups, which is a major part in the whole Langlands program.

4 Motivating Characters

The classification of all irreducible representations for abelian groups is a great success, thanks to Schur's Lemma. However, we will see that life is nowhere near as nice for non-abelian groups. We demonstrate this by computing the irreducible representations of the non-abelian group of smallest order: the symmetric group S_3.

Let V be some representation of S_3. We will restrict our action first to the cyclic group $C_3 < S_3$, then observe how transpositions act on V. Because C_3 is abelian, we know that for every $v \in V$, the group C_3 acts as a homothety. Even better, the homothety must be of order 3. Letting $\omega = e^{2\pi i/3}$ and $\sigma = (1\,2\,3)$, we have $\sigma(v) = \omega^i v$ for some $i \in \mathbb{Z}/3\mathbb{Z}$.

Now let $\tau = (1\,2)$. Note that σ and τ generate S_3, so we just need to understand the action of τ on a given $v \in V$. Assuming v is a σ-eigenvector with eigenvalue ω^i and using the relation $\sigma\tau = \tau\sigma^2$, we get

$$\sigma(\tau \cdot v) = \tau(\sigma^2 \cdot v) = \tau(\omega^{2i} v) = \omega^{2i}\tau(v).$$

We now proceed with casework. First, suppose that $\sigma v = \omega^i v$ for $i \neq 0$. Then, τv satisfies $\sigma(\tau v) = \omega^{2i}\tau(v)$ from above. Since $v, \tau v$ have different σ-eigenvalues, they are linearly independent. However, note that $\operatorname{span}\{v, \tau v\}$ is invariant under both σ and τ, so we have found a dimension-2 irreducible representation.

Exercise 4.1. Show that this representation is isomorphic to the *standard representation*, Exercise 3.3.

Now suppose $\sigma v = v$. We have that either τv is linearly dependent or independent to v. If τv is linearly dependent to v, then $\tau^2 = 1$ dictates that either $\tau v = v$ or $\tau v = -v$. If the former is true, then (v) is a *trivial representation*; the latter gives (v) as the *alternating representation.*

Finally, suppose v and τv are linearly independent. We can show that $\operatorname{span}\{v, \tau v\}$ decomposes into the trivial and alternating representations. We can see this by noting $v + \tau v$ and $v - \tau v$ are both trivial under σ and satisfy

$$\tau \cdot (v + \tau v) = v + \tau v, \qquad \tau \cdot (v - \tau v) = -(v - \tau v),$$

so $(v + \tau v)$ gives the trivial representation and $(v - \tau v)$ gives the alternating representation. As we have exhausted all cases, we can conclude that **the only irreducible representations of S_3 are the trivial, alternating, and standard representations**.

But if I asked you to find the irreducible representations of S_n for $n > 3$, then we seem to be at a loss. Given a representation V of S_n, the representation of the cyclic subgroup $C_n < S_n$ given by V now introduces n different eigenvalues we need to keep track of. This seems highly inefficient, if not intractable.

The glimpse of hope is that the eigenvalues was not only our main crutch to identifying all irreducible representations of S_3, but it also tells us how to decompose an arbitrary representation. More specifically, if U, U', W are the trivial, alternating, and standard representations, respectively, then the values (a, b, c) for a representation $V = U^{\oplus a} \oplus U'^{\oplus b} \oplus W^{\oplus c}$ can be computed from the following values. In the table, the value corresponding to (g, α) corresponds to the number of linearly independent g-eigenvectors with eigenvalue α.

$(\sigma, 1)$	(σ, ω)	(σ, ω^2)	$(\tau, 1)$	$(\tau, -1)$
$a+b$	c	c	$a+c$	$b+c$

So our hope is to track the eigenvalues and their corresponding eigenvectors for any g-action. This, as we have established, is clunky. But what happens if we took the sum of the eigenvalues – in other words, what happens if we take the trace of any g-action? To our great joy, we are able to retain all information from the trace. This motivates our definition for the character of a representation.

5 Characters

As outlined above, we will consider the trace of every g-action. This gives us a complex-valued function on G, which we call our character.

Definition 5.1. Given a representation (V, ρ) of G, the **character** of V is the complex-valued function χ_V on G given by

$$\chi_V(g) = \operatorname{Tr}(\rho_g).$$

To get a hold of this new definition, entertain yourself with the following exercises. (There is not much to work with at the moment, so these should basically follow from the definition.)

Exercise 5.2. Prove the following properties:

$$\chi_V(e) = \dim V, \qquad \chi_V(g^{-1}) = \overline{\chi_V(g)}, \qquad \chi_V(ghg^{-1}) = \chi_V(h).$$

Remark 5.3. Note that $\chi_V(ghg^{-1}) = \chi_V(h)$ implies that these characters are invariant on conjugacy classes. We call such functions as **class functions** on G. It turns out that all class functions on G are just characters of some representation of G.

Exercise 5.4. Prove that $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$ and $\chi_{V\otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$.

Remark 5.5. I am obligated to make a remark that these characters, despite their name, are *not* like the characters in group theory, i.e., group homomorphisms $\phi : G \to \mathbb{C}^\times$. The group theory characters are actually just one-dimensional complex representations of G, while characters in representation theory do not exhibit the same multiplicative structure. As an exercise, verify that for general $g, h \in G$, we do not necessarily have the equality $\chi_V(g)\chi_V(h) = \chi_V(gh)$.

Exercise 5.6. Prove that $\chi_{V^\vee}(g) = \overline{\chi_V(g)}$.

Exercise 5.7. Let a finite group G act on a finite set X, and let V be the corresponding permutation representation. (Recall then that $\dim V = |X|$.) What does $\chi_V(g)$ represent?

So now we have consolidated the information of a representation into this complex-valued class function on G. How much of a representation are we able to recover from its character? We will show that **any representation can be uniquely determined by its character**. This, in fact, turns out to be not a hard task at all.

Recall that Schur's Lemma allows us to differentiate irreducible representations from each other in a very powerful way. Let us harness its power. Schur's Lemma tells us that for any two irreducible representations V and W of G,

$$\dim \mathrm{Hom}(V, W) = \begin{cases} 1 & \text{if } V \simeq W \\ 0 & \text{if } V \not\simeq W \end{cases}.$$

Suppose $V \simeq W$, and consider the character $\chi_{\mathrm{Hom}(V,V)}$. All $\phi \in \mathrm{Hom}(V, V)$ are just homotheties $\phi = \lambda$, so they are automatically G-equivariant by linearity. This means that $g \cdot \phi = \phi$, or more explicitly, seeing Example 2.14, the action is $g \cdot \phi = g \circ \phi \circ g^{-1} = g\lambda g^{-1} = \lambda = \phi$, giving $\chi_{\mathrm{Hom}(V,V)} = 1$ always. Thus, our average is

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,V)}(g) = 1. \tag{1}$$

But Example 2.14 gives us the nice isomorphism of representations $\mathrm{Hom}(V, W) \simeq V^\vee \otimes W$. Furthermore, Exercises 5.4 ad 5.6 tells us that $\chi_{V^\vee \otimes W}(g) = \overline{\chi_V(g)} \cdot \chi_W(g)$. Thus, we can reinterpret Equation 1 as

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g) = \begin{cases} 1 & \text{if } V \simeq W \\ 0 & \text{if } V \not\simeq W \end{cases}. \tag{2}$$

This gives us a recipe for an inner product. Note (Remark 5.3) that characters are class functions, which form a vector space. Denote $\mathbb{C}_{\text{class}}(G)$ as the space of class functions on G. Define an inner product on $\mathbb{C}_{\text{class}}(G)$ by

$$\langle \varphi, \psi \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g).$$

It is easy to see that this is bilinear.

Equation 2 now tells us that *the irreducible characters are orthonormal to each other* with respect to this inner product. In particular, they are linearly independent.

Corollary 5.8. *A representation is uniquely determined by its character.*

Proof. Suppose $V = \bigoplus V_i^{\oplus a_i}$ and $W = \bigoplus W_j^{\oplus b_j}$ satisfy $\chi_V = \chi_W$. By Exercise 5.4, we can decompose the characters as

$$\chi_V = \sum_i a_i \chi_{V_i}, \qquad \chi_W = \sum_j b_j \chi_{W_j}.$$

But since all the χ_{V_i}'s, χ_{W_j}'s are linearly independent, so $\chi_V = \chi_W$ can only be possible if $V_i \simeq W_i$ and $a_i = b_i$, which forces $V \simeq W$, as desired. □

Remark 5.9. Sometimes, we will use $\langle V, W \rangle$ as notation for $\langle \chi_V, \chi_W \rangle$ for convenience.

The inner product also gives us a technique to determine the multiplicity of an irreducible representation in a given representation. All you have to do is take the inner product!

Corollary 5.10. *Let V_1 be an irreducible representation. The inner product $\langle V_1, V \rangle$ gives the multiplicity of V_1 in V.*

Proof. Using the same decomposition for V as in the above proof, we have

$$\langle V_1, V \rangle = \sum_i a_i \langle V_1, V_i \rangle = a_1,$$

where the last equality follows again from Equation 2. □

It is impressive just how much information we can glean from characters alongside their inner product. This is a testament to how much information is held in the trace of a linear map and its practical convenience.

The linear independence of the irreducible characters means that there can only be finitely many irreducible representations for a finite group G.

Corollary 5.11. *Let m be the number of conjugacy classes in G. There are at most m irreducible representations of G.*

Remark 5.12. This will immediately be made obsolete by Corollary 5.14, which asserts equality.

Proof. Let $C_1, C_2, \ldots, C_m$ be the m conjugacy classes of G. Consider the class functions

$$f_i = \begin{cases} 1 & \text{on } C_i \\ 0 & \text{on } C_{j \neq i} \end{cases}.$$

Clearly, these are linearly independent and span all of $\mathbb{C}_{\text{class}}(G)$. But we know the irreducible characters $\{\chi_i\}$ are linearly independent in $\mathbb{C}_{\text{class}}(G)$, so $\#\text{irreps} = \#\{\chi_i\} \leq m$, as desired. □

We will now show equality holds.

Proposition 5.13. The irreducible characters of G form an orthonormal basis for the space of class functions $\mathbb{C}_{\text{class}}(G)$ on G.

Proof. Let $\chi_1, \ldots, \chi_n$ be the irreducible characters of G, and suppose $f \in \mathbb{C}_{\text{class}}(G)$ satisfies $\langle f, \chi_i \rangle = 0$ for all $1 \leq i \leq n$. We wish to show $f \equiv 0$.

We will employ another averaging trick in order to invoke Schur's Lemma. Let (V, ρ) be an arbitrary representation of G. Consider the endomorphism of V given by

$$f_0 = \sum_{g \in G} \overline{f(g)} \rho_g.$$

We will show that this is G-equivariant, which would force f_0 to be a homothety. Let $h \in G$. We can compute

$$\begin{aligned} f_0 \circ \rho_h &= \sum_{g \in G} \overline{f(g)} \rho_{gh} \\ &= \sum_{g \in G} \overline{f(hgh^{-1})} \rho_{hg} \\ &= \sum_{g \in G} \overline{f(g)} \rho_h \rho_g \\ &= \rho_h \circ \sum_{g \in G} \overline{f(g)} \rho_g \\ &= \rho_h \circ f_0. \end{aligned}$$

This means $f_0 = \lambda$ by Schur's Lemma. We can explicitly compute λ via

$$\lambda = \frac{\operatorname{Tr}(f_0)}{\dim V} = \frac{1}{\dim V} \sum_{g \in G} \overline{f(g)} \chi_V(g) = \frac{|G|}{\dim V} \langle f, \chi \rangle .$$

By our assumption $\langle f, \chi_i \rangle$ for all irreducible χ_i, it follows from bilinearity of the inner product that $\lambda = 0$, so $f_0 \equiv 0$.

Now we consider (V, ρ) to be the regular representation of G. For any basis element e_h of G, we have

$$0 = f_0(e_h) = \sum_{g \in G} \overline{f(g)} \rho_g(e_h) = \sum_{g \in G} \overline{f(g)} e_{gh}.$$

This can only be possible if $\overline{f(g)} = 0$ for all $g \in G$, from which we conclude $f \equiv 0$ as desired. □

Corollary 5.14. *The number of irreducible representations of G agrees with the number of conjugacy classes in G.*

χ	C_1	C_2	$\cdots$	C_m
V_1	α_{11}	α_{12}	$\cdots$	α_{1m}
V_2	α_{21}	α_{22}	$\cdots$	α_{2m}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_m	α_{m1}	α_{m2}	$\cdots$	α_{mm}

Now that we know exactly how many irreducible representations exist for a group, we can construct a group's **character table**, which gives the values of a given irreducible character at every conjugacy class. Denoting $C_1, \dots, C_m$ as the conjugacy classes of G and $V_1, \dots, V_m$ as the irreducible representations, we can make the following table:

Denote χ_i as the character of representation V_i. By orthogonality of irreducible characters, we have for any $i \neq j$,

$$\langle \chi_i, \chi_j \rangle = \sum_{\ell=1}^{m} |C_\ell| \alpha_{i\ell} \overline{\alpha_{j\ell}} = 0.$$

In this spirit, define $A = (\sqrt{|C_j|}\alpha_{ij})_{i,j}$ and $B = (\sqrt{|C_i|}\overline{\alpha_{ji}})_{i,j}$. The orthogonality condition dictates that $AB^\intercal = I$. Taking the transpose of both sides gives $BA^\intercal = I$, which gives us **orthogonality of the columns**. Written explicitly, we have proved the following proposition using only a simple fact from linear algebra. I find this a really nice proof!

Proposition 5.15 (Orthogonality of Columns)**.** Let α_{ij} represent the values of the irreducible characters as above. Then,

$$\sum_{\ell=1}^{m} \alpha_{\ell i} \overline{\alpha_{\ell j}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

6 Irreducible Characters of S_4

We will now apply our techniques to construct the character table for S_3 and S_4, before tackling S_5 in the next section.

But before that, I want to lay out a few more notable facts which can be obtained by applying our results from the previous section to the regular representation of a group G.

Proposition 6.1. Fix a finite group G. The regular representation V_R of G contains all irreducible representations of G, with any irreducible representation V having multiplicity $\dim V$.

Proof. Computing the trace on the regular representation is straightforward: letting

χ_R denote the character of V_R, we have

$$\chi_R(g) = \begin{cases} \dim V_R = |G| & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}.$$

Using Corollary 5.10, we obtain the multiplicity of any irreducible V_i of G in V_R is

$$\langle \chi_i, \chi_R \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \chi_R(g) = \frac{1}{|G|} \overline{\chi_i(e)} |G| = \dim V_i,$$

as desired. □

Corollary 6.2. *Let $V_1, \ldots, V_n$ be the irreducible representations of G. Then*

$$|G| = \sum_{i=1}^{n} (\dim V_i)^2.$$

Proof. This follows directly from the decomposition of V_R via Proposition 6.1. □

Now we are prepared to look at the irreducible representations of S_3. For now, we will take the following fact for granted, as this is not a seminar on combinatorics:

Fact 6.3. *The conjugacy classes of S_n are determined by the different cycle types, e.g., a product of disjoint n_i-cycles where $n_1 + \cdots + n_r = |S_n|$.*

The crux of the proof is that the conjugate of any ℓ-cycle is still an ℓ-cycle: if $\sigma = (a_1\, a_2\, \cdots\, a_\ell)$ and $\tau \in S_n$, then $\tau\sigma\tau^{-1} = (\tau(a_1)\, \tau(a_2)\, \cdots\, \tau(a_\ell))$. This is left as an exercise for the reader.

Example 6.4. The cycle types of S_3 are the 3-cycles, transpositions (product of a 2-cycle with a 1-cycle), and the identity (product of all 1-cycles). For S_n, the cycle types are the 4-cycles, the 3-cycles, the product of two disjoint 2-cycles, transpositions, and the identity. One can observe that the cycle types are determined by partitions of n; consequently, the number of irreducible representations of S_n is equal to the number of partitions of n.

As said above, there are three cycle types of S_3, so there are three irreducible representations. Call them V_1, V_2, V_3. By Corollary 6.2, we have $(\dim V_1)^2 + (\dim V_2)^2 + (\dim V_3)^2 = 6$. This is only possible if the dimensions are $1, 1, 2$ in some order. We know S_3 admits the trivial and alternating representations (call U and U'), so we are left with just an irreducible representation of dimension 2.

Consider the standard representation W from Example 3.3. If it is not irreducible, then its character must have a nonzero inner product with either the trivial or alternating characters. Let σ and τ be any 3-cycle and transposition, respectively. Simple computations (or rather, realizing the standard representation as the complement of the trivial representation in the permutation representation) gives

$$\begin{aligned}(\chi_W(e), \chi_W(\tau), \chi_W(\sigma)) &= (\chi_{\mathbb{C}^3}(e), \chi_{\mathbb{C}^3}(\tau), \chi_{\mathbb{C}^3}(\sigma)) - (\chi_U(e), \chi_U(\tau), \chi_U(\sigma)) \\ &= (3, 1, 0) - (1, 1, 1) \\ &= (2, 0, -1).\end{aligned}$$

We can now check, using the fact that there are three distinct transpositions and two distinct 3-cycles,

$$\begin{aligned}\langle \chi_U, \chi_W \rangle &= \frac{1}{6}\left(\overline{\chi_U(e)}\chi_W(e) + 3\overline{\chi_U(\tau)}\chi_W(\tau) + 2\overline{\chi_U(\sigma)}\chi_W(\sigma)\right) \\ &= \frac{1}{6}\left(1 \cdot 2 + 3 \cdot 0 + 2 \cdot -1\right) = 0, \\ \langle \chi_{U'}, \chi_W \rangle &= \frac{1}{6}\left(\overline{\chi_{U'}(e)}\chi_W(e) + 3\overline{\chi_{U'}(\tau)}\chi_W(\tau) + 2\overline{\chi_{U'}(\sigma)}\chi_W(\sigma)\right) \\ &= \frac{1}{6}(1 \cdot 2 + 3 \cdot -1 \cdot 0 + 2 \cdot -1) = 0,\end{aligned}$$

so W is irreducible and we have a character table

χ	id	$(1\,2)$	$(1\,2\,3)$
U	1	1	1
U'	1	-1	1
W	2	0	-1

We can do a similar thing for S_4. There are five partitions of $n = 4$, so there are five different conjugacy classes, given by the representatives id, $(1\,2), (1\,2\,3), (1\,2)(3\,4)$, and $(1\,2\,3\,4)$. We start again with the trivial representation and the alternating representation, which are both guaranteed to be irreducible. We can also test if the standard representation is irreducible: the character values of the permutation representation on each conjugacy class, in order listed above, is $(4, 2, 1, 0, 0)$. Subtracting off the character values for the trivial representation gives us the values $(3, 1, 0, -1, -1)$ for the standard representation W. Taking $\langle \chi_W, \chi_W \rangle$ gives

$$\langle \chi_W, \chi_W \rangle = \frac{1}{24}\left(3 \cdot 3 + \binom{4}{2} \cdot 1 \cdot 1 + 3 \cdot -1 \cdot -1 + 3! \cdot -1 \cdot -1\right) = 1,$$

so indeed W is irreducible.

We use Corollary 6.2 to determine the dimensions of the other two irreducible representations. Let the missing dimensions be d_1 and d_2. We require

$$24 = 1^2 + 1^2 + 3^2 + d_1^2 + d_2^2,$$

which is only possible when $(d_1, d_2) = (2, 3)$ in some order. We have a guess for the other dimension 3 representation: $U' \otimes W$. The character values of this representation is $(3, -1, 0, -1, 1)$; note that $\chi_{U' \otimes W}(g)^2 = \chi_W(g)^2$ for all $g \in G$, which means

$\langle \chi_{U'\otimes W}, \chi_{U'\otimes W} \rangle = \langle \chi_W, \chi_W \rangle = 1$, affirming $U' \otimes W$ as irreducible. Our character table now looks like

χ	id	$(1\,2)$	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4)$
U	1	1	1	1	1
U'	1	-1	1	1	-1
V	2				
W	3	1	0	-1	-1
$U' \otimes W$	3	-1	0	-1	1

The rest can be filled in via the orthogonality of columns (Proposition 5.15). We will leave this as an exercise to the reader.

It's great that we're able to determine all irreducible characters, as any representation is uniquely determined by its character, but knowing its values doesn't really tell us much as to what the representation actually is. We have a strong understanding of the trivial, sign, and standard representations (U, U', and W). We can realize $U' \otimes W$ quite concretely from this (as a vector space, it is isomorphic to W, but the action now takes into account the permutation sign).

For the last irreducible representation V, note the similarity in its character values with the character values of the standard representation of S_3. We can then realize the representation V as extending the standard representation of S_3 to S_4 via the projection $S_4 \to S_3$, which has kernel isomorphic to the Klein 4-group V_4. One can check that the character values of this representation matches the values in the third row of the table, once the reader fills it in.

7 S_5 and other tidbits

Finally, we can do the same for irreducible representations of S_5. This time, there are seven partitions of $n = 5$. We have as irreducible representations, once again, the trivial (U), alternating (U'), standard (W), and the tensor $U' \otimes W$. There are three remaining irreducible representations – call their respective dimensions d_1, d_2, d_3. Corollary 6.2 tells us

$$120 = 1^2 + 1^2 + 4^2 + 4^2 + d_1^2 + d_2^2 + d_3^2 \implies d_1^2 + d_2^2 + d_3^2 = 86,$$

which is only possible when $\{d_1, d_2, d_3\}$ is $\{7, 6, 1\}$ or $\{6, 5, 5\}$. But we have expired all possible dimension 1 representations (namely, U and U'), so the dimensions must take on the latter values.

To construct the dimension 6 irreducible representation, we will decompose the tensor product representation $W \otimes W$, where W is the standard representation as before. As a brief interlude, we introduce the following fact that helps motivate this move:

Proposition 7.1. Let V be a faithful representation of G. Then, for every irreducible representation W, there exists some $n \in \mathbb{N}$ such that W is a subrepresentation of $V^{\otimes n}$.

Proof. There are two super slick proofs of this. The first is from Fulton-Harris, while the second is attributed to our very own David Speyer. They are basically the same proof, just worded slightly differently.

Proof 1: We will denote $a_n = \langle \chi_W, \chi_V^n \rangle$. Consider the series

$$f(z) = \sum_{n \geq 0} a_n z^n.$$

Let the conjugacy classes of G be $C_1, \ldots, C_m$. Invoking the definition, we have

$$\begin{aligned} f(z) = \sum_{n \geq 0} a_n z^n &= \sum_{n \geq 0} \langle \chi_W, \chi_V^n \rangle \, z^n \\ &= \sum_{n \geq 0} \sum_{i=1}^{m} |C_i| \overline{\chi_W(C_i)} \chi_V(C_i)^n z^n \\ &= \sum_{i=1}^{m} \frac{|C_i| \cdot \overline{\chi_W(C_i)}}{1 - \chi_V(C_i) z}. \end{aligned}$$

If $f \equiv 0$, then we require

$$\sum_{C \neq \{e\}} \frac{|C| \cdot \overline{\chi_W(C)}}{1 - \chi_V(C) z} = -\frac{\dim(W)}{1 - (\dim V) z}.$$

But each $(1 - \chi_V(C_i) z)$ are irreducible in $\mathbb{C}[z]$, and $|\chi_V(C_i)| \leq \chi_V(e) = \dim V$, with equality if and only if $C_i = \{e\}$ by faithfulness. As none of the denominators on the left hand side match the denominator on the right hand side, this equality is impossible, meaning f does not vanish everywhere. It follows that $a_n \neq 0$ for some n, as desired.

Proof 2: Denote 1 as the trivial representation. It suffices to show W is a subrepresentation of $(V \oplus 1)^{\otimes n}$ for some n, as we can use distributivity on $(V \oplus 1)^{\otimes n}$ and W must belong in strictly one of the direct summands by irreducibility. We want to show

$$\left\langle \chi_W, \chi_{(V \oplus 1)^{\otimes n}} \right\rangle \overset{?}{\geq} 1.$$

Expanding the left, we want to study

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \chi_{(V \oplus 1)^{\otimes n}(g)} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} (\chi_V(g) + 1)^n.$$

Like we established above, we have $|\chi_V(g)| \leq \chi_V(e) = \dim V$ with equality if and only if $g = e$ by faithfulness. Additionally, $|\chi_W(g)| \leq \dim W$, with equality iff $g = e$. Combining, we determine $|\chi_V(g) + 1|^n \leq (1 + \dim V)^n$ with equality iff $g = e$, so for large enough n, the sum on the right must be positive. The conclusion follows. □

Returning back to the case $n = 2$, we can decompose $V \otimes V$ into two useful representations which are significant linear algebraic subspaces:

$$V \otimes V = \mathrm{Sym}^2(V) \oplus \bigwedge^2(V).$$

Letting $V = \mathrm{span}\{e_i\}_i$, recall that $\mathrm{Sym}^2(V)$ has basis consisting of all $\frac{e_i \otimes e_j + e_j \otimes e_i}{2}$ for $i \leq j$, while $\bigwedge^2$ has basis consisting of $\frac{e_i \otimes e_j - e_j \otimes e_i}{2}$. One can see that these are both subrepresentations. Furthermore, denoting $n = \dim V$, it is easy to compute that

$$\dim \mathrm{Sym}^2(V) = \binom{n+1}{2}, \qquad \dim \bigwedge^2(V) = \binom{n}{2}.$$

Exercise 7.2. Verify that the characters of $\mathrm{Sym}^2(V)$ and $\bigwedge^2(V)$ satisfy

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right)$$
$$\chi_{\bigwedge^2(V)}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

(Just follow the definition and do some simple algebraic manipulations.)

This sets us up nicely: given our standard representation W, we have $\dim \bigwedge^2 W = \binom{4}{2} = 6$, so it is a strong candidate for the irreducible representation with dimension 6. Using the above exercise, we can explicitly compute its character, filled out in the table below.

χ	id	$(1\,2)$	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4)$	$(1\,2\,3\,4\,5)$	$(1\,2)(3\,4\,5)$
U	1	1	1	1	1	1	1
U'	1	-1	1	1	-1	1	-1
W	4	2	1	0	0	-1	-1
$U' \otimes W$	4	-2	1	0	0	-1	1
V_1	5						
V_2	5						
$\bigwedge^2 W$	6	0	0	-2	0	1	0

Note that if $\langle \chi_{V_1}, \chi_{V_1} \rangle = 1$, then $\langle \chi_{U' \otimes V_1}, \chi_{U' \otimes V_1} \rangle = 1$ since $\chi_{U'}^2 = 1$, so we can take $V_2 = U' \otimes V_1$. Given this, we can invoke the orthogonality of columns (Proposition 5.15) to fill out the rest of the columns. The resulting table looks like this:

χ	id	$(1\,2)$	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4)$	$(1\,2\,3\,4\,5)$	$(1\,2)(3\,4\,5)$
U	1	1	1	1	1	1	1
U'	1	-1	1	1	-1	1	-1
W	4	2	1	0	0	-1	-1
$U' \otimes W$	4	-2	1	0	0	-1	1
V	5	1	-1	1	-1	0	1
$U' \otimes V$	5	-1	-1	1	1	0	-1
$\bigwedge^2 W$	6	0	0	-2	0	1	0

Like before, we still don't have an understanding of V. The way we could access this representation is through looking at the other component $\operatorname{Sym}^2 W$ of $W \otimes W$. Computing the character of $\operatorname{Sym}^2(W)$ from Exercise 7.2, the character values in order of the conjugacy classes in the above table are $(10, 4, 1, 0, 0, 0, 1)$.

We can realize three subrepresentations of $\operatorname{Sym}^2(W)$. The first comes from obtaining the permutation representation $\mathbb{C}^5$ as a subrepresentation of $\operatorname{Sym}^2(W)$. Seeing W as the vector space

$$W = \operatorname{span}\left\{\sum_{i=1}^{5} a_i e_i : a_1 + a_2 + a_3 + a_4 + a_5 = 0\right\},$$

consider the five elements $w_i = -5e_i + \sum_j e_j$ for $1 \leq i \leq 5$. It is clear that for $\sigma \in S_5$, the action is $\sigma \cdot w_i = w_{\sigma(i)}$. Then in $\operatorname{Sym}^2(W)$, we have

$$\sigma \cdot (w_i \otimes w_i) = w_{\sigma(i)} \otimes w_{\sigma(i)}.$$

This means the vector space spanned by the pure tensors $w_i \otimes w_i$ is a subrepresentation isomorphic to the permutation representation of S_5. We know we can decompose this into $U \otimes W$, so both $U, W \subset \operatorname{Sym}^2 W$. Let V be the representation such that $\operatorname{Sym}^2 W = U \oplus W \oplus V$. We can compute $\dim V = \dim \operatorname{Sym}^2 W - \dim U - \dim W = 10 - 1 - 4 = 5$. This gives us a very strong candidate for our dimension-5 irreducible representation. Indeed, the character of V takes on values

$$\begin{aligned}
\chi_V &= \chi_{\operatorname{Sym}^2(W)} - \chi_U - \chi_W \\
&= (5, 1, -1, 1, -1, 0, 1) \\
\implies \langle \chi_V, \chi_V \rangle &= \frac{1}{120}(1 \cdot 5^2 + 10 \cdot 1^2 + 20 \cdot (-1)^2 + 15 \cdot 1^2 + 30 \cdot (-1)^2 + 20 \cdot 1^2) \\
&= \frac{1}{120} \cdot 120 = 1,
\end{aligned}$$

so V is irreducible, and we can complete our character table.

Although this has been all fun and games up to $n = 5$, one can quickly see that our current methods do not scale well. The partition numbers are exponential! There are just too many possibilities.

We won't go through the general approach to classify irreducible representations of S_n, but there is a neat combinatorial interpretation which I will briefly describe.

We've established a bijection between irreducible representations of S_n and partitions of n, with the number of conjugacy classes being the middle-man in this bijection. We will remove the middle-man, thereby enriching the bijection and using it more than just determining the number of irreducible representations.

Partitions have a nice pictorial representation in the form of a Young tableau. For instance the partition $9 = 4 + 2 + 2 + 1$ corresponds to the following Young tableau:

1	2	3	4
5	6		
7	8		
9			

There is a procedure which allows us to produce an irreducible representation from each Young tableau; as Young tableaus are equivalent to partitions, this produces all irreducible representations of S_n. We will not discuss the procedure here, but it is not too bad and the interested should check out §4.1-2 of Fulton-Harris.

Here is one nice manifestation of this explicit correspondence. Given a Young tableau, we can construct another Young tableau by flipping it along its diagonal, i.e., swapping the rows and columns. For instance, the "conjugate" of the above Young tableau given by transposing the diagram is the tableau below:

1	2	3	4
5	6	7	
8			
9			

If V is the irreducible representation associated with a given Young tableau, then the irreducible representation associated with the conjugate tableau is precisely $U' \otimes V$, where U' is the alternating representation. We proved in our computations for S_5 that tensoring by U' preserves irreducibility, and in each of the examples $n = 3, 4, 5$, exactly one irreducible representation did not have a "conjugate" irreducible.

Exercise 7.3. For each $n = 3, 4, 5$, find the irreducible representation V without a conjugate, check that $U' \otimes V = V$, and determine its corresponding Young tableau.

The **Frobenius formula** then allows us to explicitly compute the character values of an irreducible representation associated to a given Young tableau. To state the formula, we need to set up some variables. Fix a Young tableau T. Let r be the number of rows in T and let λ_i denote the number of cells in the i^{th} row of T. For $1 \leq i \leq r$, define $m_i = \lambda_i + r - i$.

We will also define two multivariate polynomials. Define the power sum $P_s(x) = P_s(x_1, \ldots, x_r)$ and the discriminant $\Delta(x) = \Delta(x_1, \ldots, x_r)$ as

$$P_s(x) = \sum_{i=1}^{r} x_i^s, \qquad \Delta(x) = \prod_{1 \leq i < j \leq r} (x_i - x_j).$$

(For our purposes, $1 \leq s \leq n$.) Finally, for any polynomial $f(x) = f(x_1, \ldots, x_r)$, denote $[f(x)]_{(a_1, \ldots, a_r)}$ as the coefficient of the $x_1^{a_1} \cdots x_r^{a_r}$-term in f.

Theorem 7.4 (Frobenius Formula)**.** Let C denote the conjugacy class of permutations in S_n expressible as the disjoint product of d_ℓ ℓ-cycles for $1 \le \ell \le n$. Let χ_T be the character of the irreducible representation corresponding to the Young tableau T. Using the notation defined above,

$$\chi_T(C) = \left[\Delta(x) \cdot \prod_{\ell=1}^{n} P_\ell(x)^{d_\ell}\right]_{(m_1,\ldots,m_r)}.$$

Example 7.5. The irreducible representation $\bigwedge^2 W$ of S_5 (where W is the standard representation of S_5) is its own conjugate, and the only Young tableau with 5 cells which is symmetric across the diagonal is the following L-shape:

1	2	3
4		
5		

We can collect the necessary values:

$$\begin{cases} r = 3 \\ (\lambda_1, \lambda_2, \lambda_3) = (3, 1, 1) \\ (m_1, m_2, m_3) = (5, 2, 1) \end{cases}.$$

Let C be the conjugacy class with representative $(1\,2)(3\,4\,5)$. Then, $d_2 = d_3 = 1$ and all other $d_\ell = 0$. The Frobenius formula tells us that

$$\begin{aligned} \chi_{\bigwedge^2 W}((1\,2)(3\,4\,5)) &= \left[(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \cdot (x_1^2 + x_2^2 + x_3^2)(x_1^3 + x_2^3 + x_3^3)\right]_{(5,2,1)} \\ &= [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]_{(0,2,1)} + [(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)]_{2,0,1} \\ &= (-1)(-1) + (-1) = 0, \end{aligned}$$

which agrees with the value from our previously computed character table.

Admittedly, this formula is not super nice, and computing the character values even for small n can be a pain. The formula, however, does give us a very nice way to obtain the dimension of χ_T without having to work so hard. The way to do this is by looking at the "hook lengths" at every cell. Given a cell in a tableau, the **hook length** at that cell is the number of cells directly below it plus the number of cells directly to its right, plus 1 (to include the cell itself). Below is the hook length of each cell for the Young tableau corresponding to $\bigwedge^2 W$ for S_5:

5	2	1
2		
1		

Theorem 7.6 (Hook Length Formula)**.** Let V_T be the irreducible representation of S_n corresponding to a Young tableau T. Denote $\lambda_1, \ldots, \lambda_n$ denote the hook lengths of the cells in T. Then,

$$\dim V_T = \frac{n!}{\lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_n}.$$

Example 7.7. Going back to the example for $\bigwedge^2 W$ (where again W is the standard representation for S_5, so $\dim W = 4$) and using the hook lengths obtained above, we can compute

$$\binom{4}{2} = \binom{\dim W}{2} = \dim \bigwedge^2 W = \frac{5!}{5 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 3!,$$

which is true.

8 Transitioning to $\mathrm{GL}_2(\mathbb{F}_q)$

To maintain sanity, we will assume $2 \nmid q$.

Representations of Lie groups is a large and rich study of its own. It is no surprise, therefore, that representations of Lie groups over fields of number theoretic interest (in particular, fields of characteristic p and non-Archimedean local fields) lend themselves to a very large story that continues to be developed. For instance, the representation theory of p-adic groups (i.e., the K-points of a reductive group for some p-adic field K) is a huge area of research.

We will be looking at matrix groups over a finite field $\mathbb{F}_q$, specifically the representations of $\mathrm{GL}_2(\mathbb{F}_q)$. One may think that nothing interesting is happening in this case, but identifying the representations of $\mathrm{GL}_2(\mathbb{F}_q)$ both is one of the only accessible examples and gives a glimpse of the more general theory. Although we won't talk about $\mathrm{SL}_2(\mathbb{F}_q)$ here, the representation theory of $\mathrm{SL}_2(\mathbb{F}_q)$ is also incredibly rich, so rich in fact that there's a whole textbook literally called "Representations of $\mathrm{SL}_2(\mathbb{F}_q)$" by Cédric Bonnafé.[1]

Our focus now can be summarized in two questions:

(Q1) What are the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$?

(Q2) What arithmetic information do they hold? In other words, why do we care?

A good place to start for (Q1) is to determine all conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$. These are determined by the eigenvalues of a given matrix, which satisfy some quadratic polynomial over $\mathbb{F}_q$ given by its characteristic polynomial. Let λ_1, λ_2 be the two eigenvalues.

[1] The first time I heard about this book, I strongly believed that mathematicians had too much free time on their hands and the book was unimportant. I realize now just how naïve I was back then.

There are four possibilities to consider:

$$1.\ \lambda_1 = \lambda_2 \in \mathbb{F}_q, \qquad 2.\ \lambda_1 \neq \lambda_2 \in \mathbb{F}_q, \qquad 3.\ \lambda_1 \neq \lambda_2 \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q.$$

The first gives rise to two conjugacy classes, given by the following representatives:

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \qquad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

There are $(q-1)$ conjugacy classes of each type, as $\lambda \in \mathbb{F}_q^\times$. The second gives us another "type" of conjugacy class:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Because $\left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{smallmatrix}\right]$, the number of such conjugacy classes is equal to $\binom{q-1}{2} = \frac{(q-1)(q-2)}{2}$.

Finally, if neither λ_1, λ_2 are in $\mathbb{F}_q$, then they must be in some quadratic extension since they satisfy some quadratic over $\mathbb{F}_q$. Let $\theta \in \mathbb{F}_q$ be a quadratic non-residue, so $\mathbb{F}_{q^2} \cong \mathbb{F}_q[\sqrt{\theta}]$. Express $\lambda_1 = a + b\sqrt{\theta}$ for $a, b \in \mathbb{F}_q$ and $b \neq 0$; this forces $\lambda_2 = a - b\sqrt{\theta}$. One can show that

$$\begin{bmatrix} a & b\theta \\ b & a \end{bmatrix} \sim \begin{bmatrix} a + b\sqrt{\theta} & 0 \\ 0 & a - b\sqrt{\theta} \end{bmatrix} \sim \begin{bmatrix} a - b\sqrt{\theta} & 0 \\ 0 & a + b\sqrt{\theta} \end{bmatrix} \sim \begin{bmatrix} a & -b\theta \\ -b & a \end{bmatrix},$$

so there are $|\mathbb{F}_q| = q$ choices for a and $\left|\mathbb{F}_q^\times\right|/2 = (q-1)/2$ choices for b, giving a total of $\frac{1}{2}(q^2 - q)$ conjugacy classes of this type.

To summarize, we have four "types" of conjugacy classes, described by the following table:

Type	I	II	III	IV
Representative	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} a & b\theta \\ b & a \end{bmatrix}$
# of conj. classes	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$
Size of class	1	q^2-1	q^2+q	q^2-q

Remark 8.1. For shorthand notation, we will denote the representatives for these conjugacy classes as

$$a(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad b(\lambda) := \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad a(\lambda_1, \lambda_2) := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad c(\alpha, \beta) := \begin{bmatrix} \alpha & \beta\theta \\ \beta & \alpha \end{bmatrix}.$$

As a sanity check, we can check the sum of the sizes of all conjugacy classes is exactly $|\operatorname{GL}_2(\mathbb{F}_q)|$. We know $|\operatorname{GL}_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$, as there are (q^2-1) choices for the top row and $(q^2 - q)$ choices for the bottom to ensure linear independence. We can compute

$$\begin{aligned}(q-1)\cdot 1 + (q-1)(q^2-1) + \frac{1}{2}(q-1)(q-2)(q^2+q) + \frac{1}{2}q(q-1)(q^2-q)\\ &= (q-1)\left(1 + (q^2-1) + \frac{1}{2}(q-2)(q^2+q) + \frac{1}{2}q(q^2-q)\right)\\ &= (q-1)\left(q^2 + \frac{q}{2}\left((q-2)(q+1) + q(q-1)\right)\right)\\ &= (q-1)\left(q^2 + q(q^2-q-1)\right)\\ &= (q-1)q(q-1)(q+1)\\ &= (q^2-1)(q^2-q)\end{aligned}$$

as expected.

Exercise 8.2. Verify the "size of class" row.

The number of conjugacy classes is just the sum of the values in the second row, which is

$$(q-1)\left(1+1+\frac{1}{2}(q-2+q)\right) = (q-1)(q+1) = q^2-1,$$

so we are now on the search for q^2-1 irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

There are some natural places we can start. For S_n, we liked to start with the permutation representation, and this gave us the standard representation. Consider the permutation representation of $\mathrm{GL}_2(\mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$, so the vector space has basis indexed by the points on $\mathbb{P}^1(\mathbb{F}_q)$. Since the action of $\mathrm{GL}_2(\mathbb{F}_q)$ on $\mathbb{P}^1(\mathbb{F}_q)$ is transitive, we have a trivial subrepresentation given by the sum of the basis elements. Subtracting the trivial subrepresentation gives another subrepresentation of dimension $|\mathbb{P}^1(\mathbb{F}_q)| - 1 = q$. This is called the **Steinberg representation**, which we denote as V or V_{St}.

We check if this is irreducible by looking at its character. Denote $\chi_{\mathbb{P}^1(\mathbb{F}_q)}$ as the character of the permutation representation and χ_U as the trivial character. Then, $\chi_V = \chi_{\mathbb{P}^1} - \chi_U$. We will compute the values of all three characters for each conjugacy class type.

Consider the representatives as in Remark 8.1, beginning with $a(\lambda)$. The points in $\mathbb{P}^1(\mathbb{F}_q)$ can be expressed as $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ for $x \in \mathbb{F}_q$ or $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. It is easy to see that $a(\lambda)$ fixes all points. On the other hand, $b(\lambda)$ only fixes $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, while $a(\lambda_1, \lambda_2)$ fixes only $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Finally, we compute

$$\begin{bmatrix} \alpha & \beta\theta \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha x + \beta\theta \\ \beta x + \alpha \end{bmatrix}, \qquad \begin{bmatrix} \alpha & \beta\theta \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

The point $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ can thus never be fixed as $\beta \neq 0$. For the point $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ to be fixed, we require $x(\beta x + \alpha) = \alpha x + \beta\theta$, or equivalently $x^2 = \theta$. But we chose θ as a quadratic non-residue in $\mathbb{F}_q$, so there are no fixed points. The values for $\chi_{\mathbb{P}^1}$ are now provided, and we can subtract the trivial character to get χ_V:

Type	I	II	III	IV
$\chi_{\mathbb{P}^1}$	$q+1$	1	2	0
χ_U	1	1	1	1
χ_V	q	0	1	-1

Using the table we made before, we can compute $\langle \chi_V, \chi_V \rangle$. We can compute

$$\begin{aligned}\sum_{g \in \mathrm{GL}_2(\mathbb{F}_q)} \overline{\chi_V(g)}\chi_V(g) &= q^2 \cdot (q-1) + 1^2 \cdot \frac{1}{2}(q-1)(q-2)(q^2+q) \\ &\quad + (-1)^2 \cdot \frac{1}{2}q(q-1)(q^2-q) \\ &= q(q-1)\left(q + \frac{1}{2}(q-2)(q+1) + \frac{1}{2}q(q-1)\right) \\ &= q(q-1)(q+q^2-q-1) \\ &= (q^2-1)(q^2-q) = |\,\mathrm{GL}_2(\mathbb{F}_q)|,\end{aligned}$$

so dividing by $|\,\mathrm{GL}_2(\mathbb{F}_q)|$ on both sides gives $1 = \langle \chi_V, \chi_V \rangle$, meaning V is irreducible. Although it is cool that we found one non-trivial irreducible representation, we need $q^2 - 1$ irreducible representations in total, so we need something better.

Although working with $G = \mathrm{GL}_2(\mathbb{F}_q)$ straight up is a bit difficult, its subgroups are easier to deal with. For instance, consider the diagonal subgroup

$$T = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{F}_q^\times \right\}.$$

Remark 8.3. T stands for maximal torus. For a general compact Lie group, the maximal torus is the maximal compact connected abelian Lie subgroup. Given a matrix group like $\mathrm{GL}_n(K)$, the maximal torus is just the diagonal subgroup, which is isomorphic to $(K^\times)^n$.

We can construct lots of representations of T, albeit they are 1-dimensional: given two characters $\chi_1, \chi_2 : \mathbb{F}_q^\times \to \mathbb{C}^\times$, we can define the character $\chi = \chi_1 \otimes \chi_2$ on T by

$$\chi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \chi_1(a)\chi_2(b).$$

Two other notable subgroups are the unipotent subgroup

$$U = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{F}_q \right\}$$

and the Borel subgroup consisting of upper triangular matrices

$$B = \left\{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} : a, b \in \mathbb{F}_q^\times, c \in \mathbb{F}_q \right\}.$$

We have $B = T \rtimes U$, as for any $\left[\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right] \in B$, we can write

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & c/a \\ 0 & 1 \end{bmatrix}.$$

Furthermore, it is easy to check that U is normal in B ($bub^{-1} \in B$ and its diagonal entries are the products of the diagonal entries of b, u, and b^{-1}, separately), so we can identify $T \simeq B/U$. Given this, we can take our character $\chi = \chi_1 \otimes \chi_2$ and extend it to B by keeping it trivial on U. Explicitly, define χ as a character on B by

$$\chi\left(\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}\right) = \chi_1(a)\chi_2(b).$$

How can we make the final jump and turn our representation of B into a representation on all of G? This is doable – we can *induce* a representation from a subgroup to the whole group – but it requires some work. It is the focus of the next section.

Before discussing induced representations, we will give a sneak peek of (Q2) by referring to $\mathrm{GL}_2(\mathbb{R})$. Somewhere in Hari's talk notes on automorphic forms, you may be able to find him considering the representation $L^2(\Gamma\backslash \mathrm{GL}_2)$ of GL_2, where $\Gamma \subset \mathrm{GL}_2$ is some subgroup with finite covolume. By the Peter-Weyl Theorem, this representation decomposes into unitary irreducible representations, which are somehow related to modular forms and Maass forms. For instance, one can explicitly associate a cuspidal eigenform with one of these irreducible representations. These irreducible representations, fittingly, are called **cuspidal representations**, and it is clear that they contain significant arithmetic information.

Our proceeding study of $\mathrm{GL}_2(\mathbb{F}_q)$ will produce a class of irreducible representations which we will also call cuspidal. Although discussing automorphic forms over a finite field is silly, these cuspidal representations are significant in their own right. For instance, consider the following neat fact:

Fact 8.4. *The number of cuspidal representations of* $\mathrm{GL}_n(\mathbb{F}_q)$ *agrees with the number of irreducible degree* n *monic polynomials in* $\mathbb{F}_q[X]$.

The finite field case is interesting because they are closely tied with the local field case. For instance, there is a way to lift a cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ to a supercuspidal representation of $\mathrm{GL}_n(K)$ for some p-adic field K. Studying representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ leads to the study of p-adic modular forms, among other things.

9 Induced Representations

Fix a (finite) group G and a subgroup $H \leq G$. Generally, V will be a representation of G and W will be a representation of H unless otherwise specified.

Given a representation V of G, we can restrict the G-action to an H-action to produce a representation of H, which we denote as $\operatorname{Res}_H^G V$. Note that this agrees with our original V as vector spaces, it just sees less of the group action.

This is clearly a loss of information. What if we were able to go in the other direction? What if, given a representation W of H, we could build up to a representation of all of G? This is the premise of induced representations.

Definition 9.1 (Induced Representation)**.** Let V be a representation of G and $W \subset V$ a representation of H. We say that V is the **induced representation from H of W** (or V is induced from W, or V is induced from H, based on what is known) if

$$V = \bigoplus_{[g] \in G/H} g \cdot W.$$

We denote V as $\operatorname{Ind}_H^G W$.

Remark 9.2. Note that this is well-defined/not dependent on choice of coset, as if $g_1 H = g_2 H$, then $g_1 = g_2 h$ for some $h \in H$, so $g_1 \cdot W = g_2 \cdot h \cdot W = g_2 \cdot W$ since W is invariant under $h \in H$.

Example 9.3 (Left Coset Action induced from Trivial)**.** Let $\mathbb{C} \cdot e_H$ be the trivial representation on H. Then, the permutation representation $V = (e_{gH})_{[g] \in G/H}$ on the cosets G/H is induced from the trivial representation, as

$$V = \bigoplus_{[g] \in G/H} g \cdot (e_H) = \bigoplus_{[g] \in G/H} e_{gH}.$$

Example 9.4 (Inducing Regular Representations)**.** Let V_H be the regular representation of H and V_G the same for G. Then, $V_G = \operatorname{Ind}_H^G V_H$. We can see this via

$$\operatorname{Ind}_H^G V_H = \bigoplus_{[g] \in G/H} [g] \cdot \bigoplus_{h \in H} e_h = \bigoplus_{g \in G} e_g = V_G.$$

Exercise 9.5. Show that $\dim \operatorname{Ind}_H^G W = [G : H] \cdot \dim W$ and $\operatorname{Ind}_K^G \operatorname{Ind}_H^K W = \operatorname{Ind}_H^G W$.

In both instances, the induced representation can be viewed as the "most natural" way to extend the H-action to all of G. This property is very nicely described by this other characterization of the induced representation.

Suppose again V is a representation of G. We can define the $\mathbb{C}$-algebra

$$\mathbb{C}[G] = \{a_1 g_1 + \cdots + a_k g_k \mid a_i \in \mathbb{C}, g_i \in G.$$

As a $\mathbb{C}$-vector space, all the group elements g_i are linearly independent, and multiplication is given by the group operation, i.e., $g_1 \cdot g_2 = g_1 g_2$ and extend linearly. Since V is a complex representation, we can extend the G-action to a $\mathbb{C}[G]$-action, making V a $\mathbb{C}[G]$-module.

Take W a representation of H, and suppose $W \subset V$ for some representation V of G. We can give W a natural $\mathbb{C}[H]$-module structure. As $\mathbb{C}[H] \subset \mathbb{C}[G]$ and both are $\mathbb{C}$-algebras, $\mathbb{C}[G]$ has an obvious $\mathbb{C}[H]$-module structure. Thus, it makes sense to upgrade the embedding $W \hookrightarrow V$ to a $\mathbb{C}[G]$-module homomorphism $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V$. When $V = \operatorname{Ind}_H^G W$, this is an isomorphism.

Proposition 9.6. Let V be a representation of G and W a representation of H such that $V = \operatorname{Ind}_H^G W$. Then, the embedding $W \hookrightarrow V$ induces an isomorphism of G-representations

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \cong V.$$

Proof. Let R be a set of representatives of G/H, so every $g \in G$ can be expressed as $g = rh$ for unique $r \in R$ and $h \in H$. We have maps

$$\begin{aligned} \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W &\simeq \bigoplus_{r \in R} rW \\ rh \otimes w &\mapsto r \cdot (hw) \\ r \otimes w &\leftarrow r \cdot w. \end{aligned}$$

These are indeed inverses of each other because $(rh) \otimes w = r \otimes (hw)$ since the tensor is over $\mathbb{C}[H]$. We can confirm that

$$\dim_{\mathbb{C}[H]} \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W = [G : H] \cdot \dim W = \dim_{\mathbb{C}[H]} V,$$

so the isomorphism is proven. □

Not only does this demonstrate the "naturality" of the action being extended to all of G, but it also guarantees both existence and uniqueness of the induced representation, which we didn't have before. Even better, the first definition starts with the parent representation V of G and identifies it as the induced representation if it satisfies the described isomorphism. The tensor product does not require us to begin with a representation of G; we can organically construct it starting from H.

We now have a construction of an induced representation, but we know virtually nothing about it. Our best hope is to somehow relate the induced representation to the original representation. What would this look like?

Denote $\operatorname{Hom}_G(V_1, V_2)$ as the representation of G (or weaker, the vector space) consisting of G-equivariant homomorphisms from V_1 to V_2.

Proposition 9.7. We have an isomorphism of vector spaces

$$\mathrm{Hom}_H(W, \mathrm{Res}_H^G V) = \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V).$$

Furthermore, this is a natural isomorphism.

Remark 9.8. For those following the category theory counselor seminar, you may notice that this defines an adjoint pair. Explicitly, if we see $\mathrm{Res}_H^G : \mathsf{Rep}(G) \to \mathsf{Rep}(H)$ and $\mathrm{Ind}_H^G : \mathsf{Rep}(H) \to \mathsf{Rep}(G)$ as (covariant) functors, then $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ are adjoint functors. This is really the adjoint pair $(- \otimes_B A, -_B)$ in disguise (where we have a ring map $B \to A$ and $-_B$ sees an A-module as a B-module via the ring map), which in turn is the Tensor-Hom adjunction in disguise.

Proof. The natural isomorphism suggests that the tensor product perspective $\mathrm{Ind}_H^G W \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ is the way to go. I guess I spoiled the punchline in the above remark: letting $B = \mathbb{C}[H]$ and $A = \mathbb{C}[G]$ with the natural inclusion map $\mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$, the $(- \otimes_B A, -_B)$ adjunction tells us immediately our desired result. I will be a little more explicit, but the below proof is the same as the proof of the invoked adjoint pair for the specific case $B = \mathbb{C}[H]$ and $A = \mathbb{C}[G]$. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, V) &\cong (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W)^\vee \otimes_{\mathbb{C}[G]} V \\
&\cong (\mathbb{C}[G]^\vee \otimes_{\mathbb{C}[H]} W^\vee) \otimes_{\mathbb{C}[G]} V \\
&\cong (W^\vee \otimes_{\mathbb{C}[H]} \mathbb{C}[G]^\vee) \otimes_{\mathbb{C}[G]} V \\
&\cong W^\vee \otimes_{\mathbb{C}[H]} (\mathbb{C}[G]^\vee \otimes_{\mathbb{C}[G]} V) \\
&\cong \mathrm{Hom}_{\mathbb{C}[H]}(W, \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V)),
\end{aligned}$$

where the last line follows from using the Tensor-Hom adjunction twice. (This is the content of Exercises 3.1-3 in the Tensor Product Problem Session notes and Proposition 7.3 in the Category Theory notes.) It now remains to prove $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V) \cong V$ as vector spaces, which one can do explicitly. (To give you a start: the forward map is $\phi \mapsto \phi(1)$.) Both are isomorphic as representations of G, and the desired result follows by restricting the action to H. □

Corollary 9.9 (Frobenius Reciprocity)**.** *We have*

$$\dim \mathrm{Hom}_H(W, \mathrm{Res}_H^G V) = \dim \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V),$$

or equivalently, denoting $\langle V_1, V_2 \rangle \coloneqq \langle \chi_{V_1}, \chi_{V_2} \rangle$,

$$\left\langle W, \mathrm{Res}_H^G V \right\rangle_H = \left\langle \mathrm{Ind}_H^G W, V \right\rangle_G .$$

This is basically the best-case scenario for what we could have. In short, if we want to see if $V \subset \mathrm{Ind}_H^G W$, we just need to compare W with $\mathrm{Res}_H^G V$. This nontrivial information about representations of G is all contained at the level of representations of H.

Remark 9.10. (Exercise in disguise) We can obtain Frobenius Reciprocity without this tensor product slickery by looking at characters from the getgo. I will omit the proof because it is just manipulating sums, but if you are interested, you should start by following the definition of the induced representation to deduce

$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_{\substack{r \in R \\ r^{-1}gr \in H}} \chi_W(r^{-1}gr) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}gs \in H}} \chi_W(s^{-1}gs),$$

where R in the first sum is some set of representatives of G/H. (Hint: consider the direct sum indexed by the left cosets G/H. When does the action by g fix a direct summand?)

We include one more interpretation of the induced representation. Although we won't use this perspective much, I find it useful because (a) it is quite tangible and (b) it hints towards a potential geometric interpretation.[2] Let (W, ρ) be a representation of $H \leq G$. Then, the induced $\operatorname{Ind}_H^G W$ can be identified as the vector space of functions

$$\{f : G \to V \mid f(hg) = \rho(h)f(g) \text{ for } h \in H\},$$

where the G-action is right translation $s \cdot f(g) := f(gs)$.

One way to see this is to realize that although a representation is in general not isomorphic to its dual, we can assert $\mathbb{C}[G] \cong \mathbb{C}[G]^\vee$ as a representation of H. We know $\chi_{\mathbb{C}[G]^\vee} = \overline{\chi_{\mathbb{C}[G]}}$, but $\chi_{\mathbb{C}[G]}(h)$ simply counts fixed points of multiplication by h in G, which is always an integer. Thus, we have equality $\chi_{\mathbb{C}[G]^\vee} = \chi_{\mathbb{C}[G]}$, so

$$\operatorname{Ind}_H^G W \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \cong \mathbb{C}[G]^\vee \otimes_{\mathbb{C}[H]} W \cong \operatorname{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W),$$

which is exactly what is being described with the above characterization.

Recall that we wish to find irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$. A strong starting place for us currently is the characters on the Borel subgroup B which are trivial on the unipotent subgroup. Take such a character $\chi = \chi_1 \otimes \chi_2$, and consider $\operatorname{Ind}_B^G \chi$. Is this irreducible? If not, how can we decompose this?

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[2] I say this because Bump uses this perspective to state a "geometric" version of Mackey's Theorem, which describes the Hom-space between two induced representations. (See Theorem 32.1 in Bump's "Lie Groups" textbook.) I had a very exciting conversation with Toyesh, the resident geometer, and we feel that the induced representation is really describing sections of some vector bundle, and the double coset business that permeates Mackey theory is trying to describe both the action on the base space and the action on the fibers (which should describe some connexion). This is not very fleshed out, but I have been very confused about induced representations for a while and this seems like an extremely promising lead. Thanks Toyesh for the enlightenment! Additionally, this definition begs for a cohomological interpretation – indeed, the book on representations of $\mathrm{SL}_2(\mathbb{F}_q)$ delves a lot into some cohomology, and Deligne-Lusztig representations are constructed via ℓ-adic cohomology. Of course, I don't really know what any of these things are, but they do exist, and I think this definition is a strong foundation for such a geometric approach.

The great arsenal that is Frobenius Reciprocity allows us to even have a chance of answering this question. If we want to compute $\left\langle \operatorname{Ind}_H^G \chi, \operatorname{Ind}_H^G \chi \right\rangle_G$, we can just compute $\left\langle \chi, \operatorname{Res}_H^G \operatorname{Ind}_H^G \chi \right\rangle_H$. The one obstruction is: what the heck does $\operatorname{Res}_H^G \operatorname{Ind}_H^G \chi$ look like?

We will answer this in the general setting (i.e., for general $H \leq G$). The answer may look disgusting at first, but after overcoming the psychological block against double cosets, it's very reasonable and in fact quite nice in our example(s) of interest.

Proposition 9.11. Let $H, K \leq G$ be two subgroups and (W, ρ) a representation of H. Let $s \in K \backslash G / H$ be a double coset representative. Denote $H^s = sHs^{-1} \cap K$, and define the representation $(W^s = W, \rho^s)$ of H^s by $\rho^s(y) := \rho(s^{-1}ys)$. Then, we have an isomorphism of K-representations

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{s \in K \backslash G / H} \operatorname{Ind}_{H^s}^K W^s.$$

Proof. The idea here is that in the decomposition

$$\operatorname{Ind}_H^G W = \bigoplus_{r \in G/H} r \cdot W,$$

almost all $r \cdot W$ are not subrepresentations, so we want to group these left-coset-indexed spaces together to form subrepresentations (of K). Following this objective, let

$$V^s = \bigoplus_{\substack{r \in G/H \\ r \in KsH}} r \cdot W.$$

Any left coset representative $r \in G/H$ belongs in exactly one double coset KsH; rewriting our direct sum by indexing with double cosets, we have

$$\operatorname{Ind}_H^G W = \bigoplus_{s \in K \backslash G / H} V^s.$$

To arrive at our desired proposition, then, we wish to exhibit the K-representation isomorphism

$$\bigoplus_{\substack{r \in G/H \\ r \in KsH}} r \cdot W = V^s \cong \operatorname{Ind}_{H^s}^K W^s.$$

The trick now is to rewrite the direct sum indexing on the left. Consider the set $\{k \cdot s \mid k \in K\}$. Every $r \in G/H$ satisfying $r \in KsH$ is equivalent modulo H to an element in the set, as $r \in KsH$ implies $r = k_r s h_r$, so $r \sim k_r s$. It now suffices to find all k such that $ks \sim s$ modulo H. But this just implies $k \in sHs^{-1}$, and as $k \in K$ by default, we see that $\{r \in G/H \mid r \in KsH\}$ is in bijection with K/H^s, where for any $r \in KsH$ there exists a unique $[k] \in K/H^s$ such that $r \sim ks$. Thus, we can write

$$\bigoplus_{\substack{r \in G/H \\ r \in KsH}} r \cdot W = \bigoplus_{k \in K/H^s} k \cdot (sW).$$

The conclusion follows from seeing $W^s \cong sW$ via the action-by-s map. □

We can now determine whether an induced representation is irreducible. Since we are taking induction and restriction between H and G, we are happy to just consider the $K = H$ case.

Proposition 9.12. Let (W, ρ) be an irreducible representation of H. Then, $\operatorname{Ind}_H^G W$ is irreducible if and only if W is irreducible and $\left\langle W^s, \operatorname{Res}_{H^s}^H W\right\rangle_{H^s} = 0$ for all $s \in G - H$.

Proof. By Frobenius Reciprocity, we know $\operatorname{Ind}_H^G W$ is irreducible if and only if

$$1 = \left\langle \operatorname{Ind}_H^G W, \operatorname{Ind}_H^G W\right\rangle_G = \left\langle W, \operatorname{Res}_H^G \operatorname{Ind}_H^G W\right\rangle_H .$$

Invoking Proposition 9.11 from above, we have

$$\begin{aligned}\left\langle W, \operatorname{Res}_H^G \operatorname{Ind}_H^G W\right\rangle_H &= \sum_{s \in H\backslash G/H} \left\langle W, \operatorname{Ind}_{H^s}^H W^s\right\rangle \\ &= \sum_{s \in H\backslash G/H} \left\langle \operatorname{Res}_{H^s}^H W, W^s\right\rangle_{H^s},\end{aligned}$$

where the last equality follows from applying Frobenius Reciprocity again.

Note that each of the terms in the sum are non-negative, and for $s = e$, the inner product is simply $\langle W, W\rangle_H \geq 1$ with equality if and only if W is irreducible. In this case, all other summands must be 0, which gives the second condition in the proposition. □

10 Principal Series Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

For this discussion, let $G = \mathrm{GL}_2(\mathbb{F}_q)$. We return to our objective of finding the $q^2 - 1$ irreducible representations of G. Our current plan is to start with characters $\chi_1, \chi_2 \in \widehat{\mathbb{F}_q^\times}$, define $\chi = \chi_1 \otimes \chi_2$ as a character on $T \simeq (\mathbb{F}_q^\times)^2$, then extend it to $B = T \rtimes U$ by making it trivial on U. We now consider the induced representation $\operatorname{Ind}_B^G \chi$, which we will also denote as $\operatorname{Ind}_B^G(\chi_1, \chi_2)$, and ask if it is irreducible.

Theorem 10.1 (Irreducibility of $\operatorname{Ind}_B^G \chi$)**.** Let $\chi = (\chi_1, \chi_2)$ be a character on B as defined above.

1. If $\chi_1 \neq \chi_2$, then $\operatorname{Ind}_B^G \chi$ is irreducible.
2. If $\chi_1 = \chi_2$, then $\operatorname{Ind}_B^G \chi$ decomposes into two irreducible representations of dimension 1 and q.

Proof. The one trick we have for determining irreducibility of a representation is computing the inner product $\left\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \right\rangle_G$. We can now invoke Frobenius Reciprocity and Proposition 9.11 to simplify

$$\begin{aligned}\left\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \right\rangle_G &= \left\langle \chi, \operatorname{Res}_B^G \operatorname{Ind}_B^G \chi \right\rangle_B \\ &= \sum_{s \in B\backslash G/B} \left\langle \chi, \operatorname{Ind}_{B^s}^B \chi^s \right\rangle_{B^s}.\end{aligned}$$

The double cosets $B\backslash G/B$ may be intimidating, but we have an exceptionally nice understanding of Borel double cosets in a reductive group via the **Bruhat decomposition**. In general, if G is a connected reductive algebraic group with a Borel subgroup B and Weyl group W,[3] then the Bruhat decomposition tells us

$$G = \bigsqcup_{w \in W} BwB.$$

In the case $G = \mathrm{GL}_n$, the Weyl group turns out to be isomorphic to S_n. This is great news, particularly in our case, because then there is only one nontrivial double coset in $B\backslash G/B$!

Exercise 10.2. Let $s = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, and verify that $\mathrm{GL}_2(\mathbb{F}_q) = B \sqcup BsB$.

For the trivial double coset $s = I_2$, we have $B^s = B$ and $\chi^s = \chi$. Since χ is one-dimensional, it is irreducible, so $\langle \chi, \chi \rangle_B = 1$. Meanwhile when $s = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, we can compute

$$s \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} s^{-1} = \begin{bmatrix} c & 0 \\ b & a \end{bmatrix},$$

so $B^s = sBs^{-1} \cap B = T$. Furthermore, for any $\operatorname{diag}(a, b) \in T$, the above computation tells us

$$s \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} s^{-1} = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix},$$

so $\chi^s\left(\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]\right) = \chi\left(\left[\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right]\right) = \chi_1(b)\chi_2(a)$, i.e., $\chi^s = \chi_2 \otimes \chi_1$. We can now proceed with our

[3] The very terse explanation: given a reductive group, one can get root data, and the simple reflections of the root data generate a group called the Weyl group. This is a generalization of the Lie group setting: given a semisimple Lie algebra, one can find a root system which spans the Lie algebra and satisfies certain properties, and the simple reflections of these root systems generate the Weyl group.

computations:

$$\begin{aligned}\left\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \right\rangle_G &= \sum_{s \in B\backslash G/B} \left\langle \chi, \operatorname{Ind}_{B^s}^B \chi^s \right\rangle_{B^s} \\ &= \langle \chi, \chi \rangle_B + \left\langle \chi, \operatorname{Ind}_T^B (\chi_2 \otimes \chi_1) \right\rangle_T \\ &= 1 + \left\langle \operatorname{Res}_T^B \chi, \chi_2 \otimes \chi_1 \right\rangle_T \\ &= 1 + \langle \chi_1 \otimes \chi_2, \chi_2 \otimes \chi_1 \rangle \\ &= 1 + \frac{1}{|T|} \sum_{a,b \in \mathbb{F}_q^\times} \overline{\chi_1(a)\chi_2(b)} \chi_2(a)\chi_1(b).\end{aligned}$$

Letting $\chi_0 \coloneqq \overline{\chi_1} \cdot \chi_2$, which is still a character on $\mathbb{F}_q^\times$, we have

$$\sum_{a,b \in \mathbb{F}_q^\times} \overline{\chi_1(a)\chi_2(b)} \chi_2(a)\chi_1(b) = \sum_{b \in \mathbb{F}_q^\times} \overline{\chi_0(b)} \sum_{a \in \mathbb{F}_q^\times} \chi_0(a).$$

If $\chi_0 \neq 1$, or equivalently $\chi_1 \neq \chi_2$, then the sum of $\chi_0(a)$ as a varies across $\mathbb{F}_q^\times$ is 0 (explanation: replace a with xa for a fixed $x \in \mathbb{F}_q^\times$). Otherwise, the sum evaluates to $|\mathbb{F}_q^\times| = (q-1)$, so the double sum is $(q-1)^2 = |T|$. To summarize,

$$\begin{aligned}\left\langle \operatorname{Ind}_B^G \chi, \operatorname{Ind}_B^G \chi \right\rangle_G &= 1 + \frac{1}{|T|} \sum_{b \in \mathbb{F}_q^\times} \overline{\chi_0(b)} \sum_{a \in \mathbb{F}_q^\times} \chi_0(a) \\ &= \begin{cases} 1 & \text{if } \chi_1 \neq \chi_2 \\ 2 & \text{if } \chi_1 = \chi_2 \end{cases}.\end{aligned}$$

The proof is done for the $\chi_1 \neq \chi_2$ case. When $\chi_1 = \chi_2$, the sum of the squares of the irreducible representations in $\operatorname{Ind}_B^G(\chi_1, \chi_1)$ is exactly 2, which is only possible if there are two irreducible components.

Let $\chi = (\chi_1, \chi_1)$. We claim that $\chi_1 \circ \det$ is a one-dimensional subrepresentation of $\operatorname{Ind}_B^G \chi$. Indeed, Frobenius Reciprocity once again gives us

$$\begin{aligned}\left\langle \chi_1 \circ \det, \operatorname{Ind}_B^G \chi \right\rangle_G &= \left\langle \operatorname{Res}_B^G(\chi_1 \circ \det), \chi \right\rangle_B \\ &= \langle \chi_1 \circ \det, \chi \rangle_B \\ &= \frac{1}{|B|} \sum_{\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] \in B} \overline{\chi_1 \circ \det \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right)} \chi \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) \\ &= \frac{1}{|B|} \sum_{\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] \in B} \overline{\chi_1(ac)} \chi_1(a)\chi_1(c) \\ &= \frac{1}{|B|} \cdot |B| = 1.\end{aligned}$$

As $\dim \operatorname{Ind}_B^G \chi = [G : B] \cdot \dim \chi = (q^2-1)(q^2-q)/(q(q-1)^2) = q+1$, the other irreducible component in $\operatorname{Ind}_B^G(\chi_1, \chi_1)$ must have dimension q, and we conclude. □

Exercise 10.3. We have produced two irreducible q-dimensional representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in two different ways. One is the Steinberg representation V_{St}, the complement of the trivial representation in the permutation representation on $\mathbb{P}^1(\mathbb{F}_q)$. The other is by inducing the trivial character $\mathrm{Ind}_B^G(1,1)$ and removing the trivial representation. Show that these two are isomorphic representations. Additionally, show that the dimension-q irreducible subrepresentations of $\mathrm{Ind}_B^G(\chi_1,\chi_1)$ are isomorphic to $(\chi_1 \circ \det) \otimes V_{\mathrm{St}}$.

Furthermore, one could run the exact same computations as above in greater generality to show that $\left\langle \mathrm{Ind}_B^G(\chi_1,\chi_2), \mathrm{Ind}_B^G(\mu_1,\mu_2)\right\rangle_G$ as 0 unless $\{\chi_1,\chi_2\} = \{\mu_1,\mu_2\}$, for which the above proof gives us the nonzero inner product values between induced representations of characters on B.

This Mackey theory work has proven far more fruitful in producing irreducible representations, but it is not sufficient. By the above proof, each $\mathrm{Ind}_B^G(\chi_1,\chi_2)$ for $\chi_1 \neq \chi_2$ is irreducible. We know $\left|\widehat{\mathbb{F}_q^\times}\right| = q-1$ (take a generator of $\mathbb{F}_q^\times$ and map it to some $(q-1)^{\mathrm{st}}$ root of unity), so this gives us $\frac{1}{2}(q-1)(q-2)$ irreducible representations of dimension $q+1$. (We divide by 2 because the order of χ_1, χ_2 does not matter.) These are called the **principal series representations**. When $\chi_1 = \chi_2$, we get a 1-dimensional and q-dimensional irreducible representation, so there are $q-1$ of each. These numbers should feel very very familiar. Compare the following two tables:

Type	I	II	III	IV
Representative	$\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}$	$\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}$	$\begin{bmatrix}a & b\theta\\ b & a\end{bmatrix}$
# of conj. classes	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$

Representation Type	$\chi_1 \circ \det$	$(\chi_1 \circ \det) \otimes V_{\mathrm{St}}$	$\mathrm{Ind}_B^G(\chi_1,\chi_2)$	???
Dimension	1	q	$q+1$	???
# of Representations	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	???

So not only is there a bijection of sets between conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ and its irreducible representations, but there seems to be an explicit correspondence between the four "types" of conjugacy classes and the four "types" of irreducible representations. (At least, this seems to be the case for the first three types.) We conjecture that there is a fourth irreducible representation type, and like how the Type IV conjugacy class comes from eigenvalues not in $\mathbb{F}_q$, we see that these representations do not come from induced Borel representations like the other three types. So where do these representations come from?

Call this fourth type of irreducible representation as **cuspidal**, following our brief exposition at the beginning of §8. From our general knowledge on representations of finite groups, we can determine the missing dimension and number of representations for this cuspidal type. The number of conjugacy classes and irreducible representations must agree, so there should be $\frac{1}{2}q(q-1)$ cuspidal representations. Letting d be the dimension of these cuspidal representations, we also know from Corollary 6.2 that

$$|\operatorname{GL}_2(\mathbb{F}_q)| = (q-1)1^2 + (q-1)q^2 + \frac{1}{2}(q-1)(q-2)(q+1)^2 + \frac{1}{2}q(q-1)d^2.$$

Working through the algebra gives you $d = q-1$. For completeness, we fill out the rest of the table:

Type	$\chi_1 \circ \det$	$(\chi_1 \circ \det) \otimes V_{\text{St}}$	$\operatorname{Ind}_B^G(\chi_1, \chi_2)$	cuspidal
Dimension	1	q	$q+1$	$q-1$
# of Reps	$q-1$	$q-1$	$\frac{1}{2}(q-1)(q-2)$	$\frac{1}{2}q(q-1)$

11 Cuspidal Representations of $\operatorname{GL}_2(\mathbb{F}_q)$

The key tool for Borel induction on the characters to generate the principal series representations was that *all characters were trivial on the unipotent.* To obtain the cuspidal representations, we will now take into account the nontrivial characters on U. Note that $U \simeq \mathbb{F}_q$ via $\left[\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right] \mapsto u$.[4]

Furthermore, we were only working with characters of $\mathbb{F}_q^\times$, whereas the Type IV conjugacy classes have eigenvalues in $\mathbb{F}_{q^2}^\times$. It may make sense, therefore, to consider characters of $\mathbb{F}_{q^2}^\times$.

In fact, this is exactly the punchline: *there is an explicit correspondence between characters on $\mathbb{F}_{q^2}^\times$ which are nontrivial on $\mathbb{F}_q^\times$ and the cuspidal representations.* We will now describe the process of going from a character $\theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ to a cuspidal representation π_θ.

Definition 11.1 (Regular Character)**.** Let θ be a character on $\mathbb{F}_{q^2}^\times$. We say θ is **regular** if $\theta^q \neq \theta$.

Our correspondence, more precisely, will construct any cuspidal representation from a regular character on $\mathbb{F}_{q^2}^\times$, and this representation will be unique up to q^{th} powers of the character. Note that our table above says there should be $\frac{1}{2}q(q-1)$ cuspidal representations, and there are $\frac{1}{2}\left|\mathbb{F}_{q^2}^\times\right| = q(q-1)/2$ regular characters, up to the q^{th} power.

We will introduce two new subgroups of $G = \operatorname{GL}_2(\mathbb{F}_q)$. First, the center $Z = Z(G)$ consists of all scalars, i.e.,

$$Z = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{F}_q^\times \right\}.$$

[4]This nontrivial property on the unipotent seems deep and very important. For instance, I have seen unipotents play a big part in doing things with automorphic forms, and reductive groups are defined via the unipotent radical. However, I have yet to really understand why the unipotent is so important – definitely some food for thought here.

We can also embed $\mathbb{F}_{q^2}^\times$ into G by fixing a basis for $\mathbb{F}_{q^2}/\mathbb{F}_q$ and considering the multiplication-by-α map, for any $\alpha \in \mathbb{F}_{q^2}^\times$, as an element of $\mathrm{GL}(\mathbb{F}_{q^2}) = \mathrm{GL}_2(\mathbb{F}_q)$. Denote the image of $\mathbb{F}_{q^2}^\times \hookrightarrow G$ as the subgroup R.

Fix a non-trivial character of U, call ψ. Note that any other character on $U \simeq \mathbb{F}_q$ can be written as $\psi_a : x \mapsto \psi(ax)$ for any $a \in \mathbb{F}_q^\times$, so our choice of ψ will prove to be inconsequential.

Let θ be a regular character of $R \simeq \mathbb{F}_{q^2}^\times$. We will now twist θ by ψ to create a character θ_ψ defined on ZU given by

$$\theta_\psi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}\right) = \theta(a)\psi(u).$$

Exercise 11.2. Let $\theta_1, \theta_2, \ldots, \theta_d$ be the distinct regular characters of $\mathbb{F}_{q^2}^\times$. Note $d = q(q-1)$, as we do not differentiate q^{th} powers. Let $a \in \mathbb{F}_q$. Verify that the sets $\{\theta_{1,\psi}, \theta_{2,\psi}, \ldots, \theta_{d,\psi}\}$ and $\{\theta_{1,\psi_a}, \theta_{2,\psi_a}, \ldots, \theta_{d,\psi_a}\}$ are the same, and that if $\theta_i = \theta_j^q$, then $\theta_{i,\psi} = \theta_{j,\psi}^q$ regardless of choice of ψ.

Before stating the main result, we provide some quick group theory computations. Observe that

$$[G : ZU] = \frac{(q^2-1)(q^2-q)}{(q-1)q} = q^2 - 1$$
$$[G : R] = \frac{(q^2-1)(q^2-q)}{q^2-1} = q^2 - q.$$

In particular, $\dim \operatorname{Ind}_{ZU}^G \theta_\psi = q^2 - 1 > q^2 - q = \dim \operatorname{Ind}_R^G \theta$. We claim that in fact $\operatorname{Ind}_R^G \theta$ is a subrepresentation of $\operatorname{Ind}_{ZU}^G \theta_\psi$, and the complement subrepresentation is an irreducible cuspidal representation. This would agree at least with our claimed dimension for these cuspidals, as $(q^2-1) - (q^2-q) = q - 1$.

Theorem 11.3. Let θ be a regular character of $R \simeq \mathbb{F}_{q^2}^\times$ and ψ a fixed non-trivial character of U. Then,

1. The induced representation $\operatorname{Ind}_R^G \theta$ is a subrepresentation of $\operatorname{Ind}_{ZU}^G \theta_\psi$.
2. The complement subrepresentation

$$\pi_\theta = \operatorname{Ind}_{ZU}^G \theta_\psi - \operatorname{Ind}_R^G \theta$$

is an irreducible representation of G with dimension $q - 1$.

Proof. We will compute the character of each induced representation explicitly by computing its values on the four types of conjugacy classes, with the help of Remark 9.10.

Type I. Let $z_\lambda = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right]$ be a representative of such a conjugacy class. Note that $z \in Z$, so it commutes with everything. Hence, we can compute

$$\begin{aligned}\chi_{ZU}(z_\lambda) &= \frac{1}{|ZU|}\sum_{s\in G}\theta(\lambda) = \frac{|G|}{|ZU|}\cdot\theta(\lambda)\\ &= \frac{(q^2-1)(q^2-q)}{(q-1)q} = (q^2-1)\theta(\lambda).\end{aligned}$$

For R, we can do something similar:

$$\chi_R(z_\lambda) = \frac{1}{|R|}\sum_{s\in G}\theta(\lambda) = \frac{|G|}{|R|}\cdot\theta(\lambda) = (q^2-q)\theta(\lambda).$$

Type II. Let $z'_\lambda = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right]$. Using the fact that for $s = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we have $s^{-1}z'_\lambda s = \left[\begin{smallmatrix} \lambda & c/a \\ 0 & \lambda \end{smallmatrix}\right]$, we can compute

$$\begin{aligned}\chi_{ZU}(z'_\lambda) &= \frac{1}{|ZU|}\sum_{\substack{s\in G\\ s^{-1}z'_\lambda s\in ZU}}\theta_\psi(s^{-1}z'_\lambda s)\\ &= \frac{1}{|ZU|}\sum_{s\in B}\theta_\psi(s^{-1}z'_\lambda s)\\ &= \frac{1}{|ZU|}\cdot(q-1)q\sum_{t\in\mathbb{F}_q^\times}\theta(\lambda)\psi(t)\\ &= \theta(\lambda)\sum_{t\in\mathbb{F}_q^\times}\psi(t) = -\theta(\lambda).\end{aligned}$$

This was a bit more laborious, but thankfully we do not need to do any work for $\chi_R(z'_\lambda)$. The reason is that conjugates of elements in R can either only be of Type I or Type IV, so the sum as in Remark 9.10 is empty.

Type III. Any conjugate of an element in either ZU or R will have the same diagonal elements, so both characters vanish on this class type.

Type IV. Identify $\mathbb{F}_{q^2} \simeq \mathbb{F}_q[\sqrt{d}]$, and let $m(\alpha,\beta) = \left[\begin{smallmatrix} \alpha & \beta d \\ \beta & \alpha \end{smallmatrix}\right]$. No conjugate of $m(\alpha,\beta)$ can be in ZU, as otherwise the characteristic polynomial of $m(\alpha,\beta)$ would have roots in $\mathbb{F}_q$. On the other hand, we have

$$\begin{aligned}\chi_R(m(\alpha,\beta)) &= \frac{1}{|R|}\sum_{\substack{s\in G\\ s^{-1}m(\alpha,\beta)s\in R}}\theta(s^{-1}m(\alpha,\beta)s)\\ &= \frac{1}{|E|}\cdot|E|\cdot\Big(\theta(\alpha+\beta\sqrt{d})+\theta^q(\alpha+\beta\sqrt{d})\Big)\\ &= \theta(\alpha+\beta\sqrt{d})+\theta^q(\alpha+\beta\sqrt{d}).\end{aligned}$$

We omit the proof of the first part for brevity, but all of the character values are provided. For the irreducibility of π_θ, we compute $\chi_{\pi_\theta} = \chi_{\mathrm{Ind}_{ZU}^G \theta_\psi} - \chi_{\mathrm{Ind}_R^G \theta}$ and take the inner product:

$$\langle \chi_{\pi_\theta}, \chi_{\pi_\theta} \rangle = \frac{1}{|G|} \left((q-1)^2 \sum_{\lambda \in \mathbb{F}_q^\times} |\theta(\lambda)|^2 + (q^2-1) \sum_{\lambda \in \mathbb{F}_q^\times} |\theta(\lambda)|^2 + \frac{1}{2}(q^2-q) \sum_{\xi \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times} |\theta(\xi) + \theta^q(\xi)|^2 \right).$$

We compute $|\theta(\xi) + \theta^q(\xi)|^2 = 2 + \theta(\xi^{q-1}) + \theta(\xi^{-q+1})$, so

$$\begin{aligned}\sum_{\xi \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times} |\theta(\xi) + \theta^q(\xi)|^2 &= 2(q^2-q) + 2\left(\sum_{\xi \in \mathbb{F}_{q^2}^\times} \theta(\xi^{q-1}) - \sum_{\lambda \in \mathbb{F}_q^\times} \theta(\lambda^{q-1}) \right) \\ &= 2(q^2-q) - 2(q-1) = 2(q-1)^2,\end{aligned}$$

and substituting appropriately indeed reduces the computation to $\langle \chi_{\pi_\theta}, \chi_{\pi_\theta} \rangle = 1$. □

Corollary 11.4. *If θ_1 and θ_2 are regular characters of R, then $\pi_{\theta_1} \cong \pi_{\theta_2}$ if and only if either $\theta_1 = \theta_2$ or $\theta_1 = \theta_2^q$.*

Proof. This also follows from the character values and doing the computation directly. □

Corollary 11.5. *Every cuspidal representation comes from a regular character θ.*

Proof. This is simply a counting argument: we have $\frac{1}{2}q(q-1)$ regular characters as determined before, and each gives rise to an irreducible $q-1$-dimensional representation. Corollary 6.2 dictates that we can have no more. □

The correspondence in the $\mathrm{GL}_2(\mathbb{F}_q)$ case between conjugacy class types and irreducible representation types, as well as the correspondence between cuspidal representations and irreducible characters on $\mathbb{F}_{q^2}$, suggests that there is something very deep going on here. I am both limited in terms of my knowledge on these topics and time available to write these notes, so this will be incredibly terse. However, these are my general impressions of what lives beyond.

- (Structure on cuspidal representations) We can generalize this Borel induction to what is known as parabolic induction. The parabolic induction, or in particular the parabolic subgroup $P = M \rtimes U$ where M is the Levi subgroup of block diagonal matrices, is a bit obscured in the $\mathrm{GL}_2(\mathbb{F}_q)$ case because parabolic agrees exactly with Borel. However, for larger $n > 2$, parabolic induction is nice because it is easier to handle than Borel induction, and more importantly, *it produces a graded ring structure on the set of cuspidal representations.* To have any ring structure on the set of cuspidal representations is impressive. Furthermore, this synthetic process is somehow akin to the Eisenstein series for automorphic representations, although I have no idea of the connection. In any case, it turns out that any irreducible representation of $\mathrm{GL}_n(\mathbb{F}_q)$ can be found as a subrepresentation of some parabolic induction of cuspidal representations.

- (Lifting to local fields) Given a cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, we have a way of lifting it to a cuspidal representation of $\mathrm{GL}_n(K)$ for some p-adic field K. (Of course, $\operatorname{char}\mathbb{F}_q = p$.) One does this by first inflating the cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ to $\mathrm{GL}_n(\mathcal{O}_K)$, then using some smooth induction to extend to all of $\mathrm{GL}_n(K)$.

- (Local Langlands Correspondence) Everything seems to be pretty isolated in the world of reductive groups or groups of Lie type, but the Local Langlands Correspondence builds a bridge between this automorphic side and Galois theory. More precisely, there exists a correspondence between smooth irreducible admissible representations of $\mathrm{GL}_n(\mathbb{C})$ and a certain kind of Galois representation called Weil-Deligne representations. These are special representations of the group generated by the preimage of the Frobenius element under the map $\mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}}$.

12 Representation Theory in characteristic p of p-groups (Guest Talk)

This is my transcribed notes from the guest counselor seminar by Emma Knight on July 10, 2024. I missed some stuff at the end, but I hope this conveys the fundamental idea that there is very little we can do in characteristic p.

For the talk, k will be a field of characteristic p, G is a finite group, and V is a finite-dimensional representation of G.

At the end of Representation Theory I, we established that for any subrepresentation $W \subseteq V$, we can find a complement subspace W_0 of W which is also a subrepresentation. This involves this "averaging" trick: take some projection $\pi : V \twoheadrightarrow W$ fixing W, then consider the new projection $\pi_0 = \frac{1}{|G|} \sum_{g \in G} \rho_g \circ \pi \circ \rho_g^{-1}$.

In characteristic p, we now run into some problems. What if p divides $|G|$? Then we no longer can divide by $|G|$ in the averaging trick. Can we still ensure a decomposition into irreducibles in this case?

Let's take an example, say $G = \langle \gamma \mid \gamma^p = e \rangle$. Let's find some representations of this. We have the trivial representation, which is already irreducible. Consider the regular representation k^p, where $\gamma \cdot (e_1, \ldots, e_p) = (e_2, \ldots, e_p, e_1)$. This has a trivial subrepresentation given by the subspace $(e_1 + e_2 + \cdots + e_p)$. In characteristic 0, we could make the subrepresentation $\{(a_1, \ldots, a_p) \mid \sum a_i = 0\}$. But this is already bad, since our trivial subrepresentation is contained in this complement. Uh oh.

Let's do a more baby example. Take $p = 2$, and define the representation k^2 of G via $\gamma \cdot (a, b) = (b, a)$. How can we get the trivial subrepresentation (a, a) from this group action? The key here is to consider all eigenvectors of γ. Since $p = 2$, the only eigenvector of γ, up to scaling, is $(1, 1)$, which corresponds to the trivial subrepresentation. So we can't even decompose this dimension 2 representation.

So *how bad is this failure to decompose*? Worded differently, this failure to decompose comes from the lack of irreducible representations, so *how few irreducible representations exist*? The answer is that everything fails, and only the trivial representation exists.

> **Theorem 12.1.** Let G be a p-group and $V \neq 0$ a finite-dimensional representation. Then, $V^G = \{v \in V \mid g \cdot v = v\} \neq 0$.

This means no matter how many trivial subrepresentations I take away from a representation, I am still left with trivial subrepresentations. In other words,

Corollary 12.2. *The only irreducible representation of G is the trivial one.*

Proof. (of theorem) We proceed by induction on k in $|G| = p^k$. When $|G| = p$, write $G = \langle \gamma \mid \gamma^p = 1 \rangle$. Let V be a representation of G. In the same spirit as the exposition, we wish to find eigenvectors of the map $\gamma \cdot -$.

We know $\gamma^p v = v$ by definition, so $(\gamma^p - I)v = 0$. This means the minimal polynomial of γ must divide $x^p - 1$. Here is the punch: in characteristic p, we have the amazing factorization $x^p - 1 = (x-1)^p$, so 1 is the only eigenvalue and so there is some v such that $\gamma v = v$. This completes the base case.

Now assume this is true for all groups of order p^{n-1}. Consider a group G of order p^n. By Sylow, there exist some subgroup $H \subset G$ such that $|H| = p^{n-1}$. Furthermore, H is normal in G. Here is a quick sketch: we have a natural action G on G/H. Since $|G/H| = p$, this induces a map $G \to S_p$. The image must have order dividing p, but the coset action is nontrivial, so $|\operatorname{Im}(G)| = p$ and so $H = \ker(G \to S_p)$ is normal.

By the inductive hypothesis, we know V^H is nonzero. Let $v \in V^H$ and $g \in G$. We wish to show $h(gv) = gv$ for all $h \in H$. But this is true by normality of H, as we can write $h(gv) = g(h'v) = gv$.

This makes V^H as a G-subrepresentation of V. The action of G on V^H factors through G/H, since V^H is trivial by the H-action by definition, so there exists some $v \in V^H$ such that $gv = v$ for all $g \in G$. This concludes the inductive step. $\square$

Instead of a p-group, let's consider a group such as $G = \mathrm{GL}_2(\mathbb{F}_p)$. What are the irreducible representations of G in characteristic p? Here are a few examples discussed:

- The representation k^2 gives the map $\mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(k)$. This is the standard representation.
- Besides the trivial representation, one-dimensional representations can arise from the determinant: $g \cdot x = \det(g)^j x$. Note that these are only distinct up to j mod p.
- Recall the two irreducible representations $V \otimes V = \mathrm{Sym}^2(V) \oplus \bigwedge^2(V)$. Letting W be the standard representation (the first bullet point), we can take the symmetric powers $\mathrm{Sym}^i(W)$. One may be concerned that this produces an infinite family of irreducible representations, but these symmetric powers stop being irreducible at $i = p$.