Introduction to p-adic Hodge theory, d'après Fontaine

"Colmez Tsinghua" learning seminar

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1 Introduction

This is the fourth talk of the "Colmez Tsinghua" learning seminar, co-organized with Zheng Wang. The name refers to the course notes of Colmez from Tsinghua [Col2004].

We initiate the study of p-adic Hodge theory. I will motivate and provide the constructions for Fontaine's period rings B_{dR}, B_{HT}, and B_{cris}; state some of the main results of p-adic Hodge theory; and define (φ, Γ)-modules.

Although not evident from this talk, the connection between p-adic Hodge theory – in other words, the study of p-adic Galois representations – and p-adic L-functions is beautifully described by Perrin-Riou's conjectural framework [PR1995]. This will be discussed in a future talk.

2 p-adic Hodge theory foundations

There are many nice references out there. I was first initiated to these ideas by [FI1980], which is a precursor to [Fon1994]. The standard reference for the "classical" version[1] of p-adic Hodge theory seems to be [BC2009]. I am also using my notes from Anna Cadoret's M2 course last year, as well as several other nice surveys [Col2019, Niz2020].

2.1 Periods and comparison isomorphisms

The name *period* ring suggests that discussing periods of varieties (more generally, motives) is a suitable starting point.

[1] Meaning before the Fargues–Fontaine curve approach.

Periods are encoded in the comparison isomorphisms between the various realizations of a motive. Let M be a (pure) motive over $\mathbb{Q}$. For the "classical" (archimedean) notion of a period, we look at the Betti-to-de Rham comparison isomorphism

$$M_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \cong M_B \otimes_{\mathbb{Q}} \mathbb{C}.$$

For example, if E is an elliptic curve over $\mathbb{Q}$, then the motive $h^1(E)(1)$ comes with the isomorphism

$$\begin{gathered} H^1_{\mathrm{dR}}(E) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{sing}}(E(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\ \omega \mapsto \left(\gamma \mapsto \int_\gamma \omega \right), \end{gathered}$$

where we have implicitly invoked Poincaré duality to view the right-hand side as the dual to singular homology.

The important thing to note is that one must extend coefficients to $\mathbb{C}$ in order for this isomorphism to work. Extending to $\mathbb{C}$ is sufficient because all integrals of the form $\int_\gamma \omega$, which we refer to as periods, are valued in $\mathbb{C}$. The fundamental example to keep in mind is the following elementary exercise from complex analysis

$$\int_{S^1} \frac{dt}{t} = 2\pi i \in \mathbb{C}.$$

To shift to non-archimedean periods, we replace the Betti realization with the p-adic étale one. We know motives also come with an étale-to-de Rham comparison isomorphism

$$M_p \otimes_{\mathbb{Q}_p} B \cong M_{\mathrm{dR}} \otimes_{\mathbb{Q}} B$$

for some large enough ring B which contains all "p-adic periods." For X a smooth proper variety over $\mathbb{Q}_p$, this isomorphism translates to

$$H^i_{\text{ét}}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B \xrightarrow{\sim} H^i_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}_p} B.$$

The ring $\mathbf{B}_{\mathrm{dR}}$, defined in the next subsection, will make the above an isomorphism.

We conclude this subsection with a discussion of what $\mathbf{B}_{\mathrm{dR}}$ could look like. At the least, it should contain a p-adic analogue t of $2\pi i$. What should t look like, and in what space could it live in?

A reasonable guess to the second question is $\mathbb{C}_p$, but we will justify that $\mathbb{C}_p$ is too small. Note that the archimedean period $2\pi i$ comes from the Betti-to-de Rham comparison for $X = \mathbb{G}_{m/\mathbb{C}}$ (dt/t is a generator of $H^1_{\mathrm{dR}}(\mathbb{G}_{m,\mathbb{C}})$ and S^1 is a generator of $H_1(\mathbb{G}_{m,\mathbb{C}}(\mathbb{C}), \mathbb{Z})$). We know that the above étale-to-de Rham isomorphism should be $G_{\mathbb{Q}_p}$-equivariant, and when $X = \mathbb{G}_{m,\mathbb{Q}_p}$, we know

$$H^1_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}_p}}, \mathbb{Q}_p) \simeq \mathbb{Q}_p(1).$$

As $H^1_{\mathrm{dR}}(\mathbb{G}_m/\mathbb{Q}_p)$ carries no Galois action, we should expect $G_{\mathbb{Q}_p}$ to act on t via the cyclotomic character in order for the isomorphism to be Galois-equivariant. In any case, the Galois action should not be trivial. But the Ax-Sen-Tate Theorem (proved by Tate in [Tat1970]) guarantees that there are no such elements in $\mathbb{C}_p$.

Theorem 2.1 (Ax-Sen-Tate)**.** *Let $K/\mathbb{Q}_p$ be a finite extension, $\mathbb{C}_K$ the p-adic completion of $\overline{K}$, and $G_K = \mathrm{Gal}(\overline{K}/K)$. Then,*

$$\mathbb{C}_K(r)^{G_K} = \begin{cases} K, & r = 0 \\ 0, & r \neq 0. \end{cases}$$

Another glaring reason why $\mathbb{C}_p$ would not work as our period ring is that, due to the ultrametricity of the p-adic absolute value, log is well-defined on $\mathbb{C}_p^\times$ once we have chosen a value for $\log p$. In particular, if ε_n is a p^n-th root of unity in $\mathbb{C}_p$, then $p^n \log \varepsilon_n = \log 1 = 0$, so $\log \varepsilon_n$.

Fontaine's first attempt of a p-adic analogue of $2\pi i$ grew from the observation that $\frac{d\varepsilon_n}{\varepsilon_n}$, seen as an element in the $\mathbb{Z}_p$-module of Kähler differentials $\Omega := \Omega_{\mathcal{O}_{\overline{\mathbb{Q}_p}}/\mathbb{Z}_p}$, is not necessarily 0, thereby salvaging the above feature. We describe some properties of this element to give a sense of the shape of t.

Theorem 2.2. *Let $(\varepsilon_n)_n$ be a compatible system of p^n-th roots of unity in $\overline{\mathbb{Q}_p}$.*

1. *For any $n \in \mathbb{N}$ and $\sigma \in G_{\mathbb{Q}_p}$, we have*

$$\frac{d\varepsilon_n}{\varepsilon_n} = p\frac{d\varepsilon_{n+1}}{\varepsilon_{n+1}}, \qquad \sigma\left(\frac{d\varepsilon_n}{\varepsilon_n}\right) = \chi(\sigma)\frac{d\varepsilon_n}{\varepsilon_n},$$

   where χ is the cyclotomic character.

2. *Let $\mathfrak{a} := \{a \in \overline{\mathbb{Q}_p} : v_p(a) \geq -1/(p-1)\}$. Then, for any $a \in \mathcal{O}_{\overline{\mathbb{Q}_p}}$ and $n \in \mathbb{N}$, the map $p^{-n}\mathcal{O}_{\overline{\mathbb{Q}_p}} \to \Omega$ where $p^{-n}a \mapsto a\frac{d\varepsilon_n}{\varepsilon_n}$ induces an isomorphism*

$$\iota : \overline{\mathbb{Q}_p}/\mathfrak{a} \xrightarrow{\sim} \Omega,$$

   where for any $\sigma \in G_{\mathbb{Q}_p}$ and $\alpha \in \overline{\mathbb{Q}_p}$,

$$\sigma(\iota(\alpha)) = \chi(\sigma)\iota(\sigma(\alpha)).$$

The takeaway is that t should (1) in some way "linearize" (i.e., a log should appear somewhere) a compatible system of p^n-th roots of unity and (2) have a cyclotomic $G_{\mathbb{Q}_p}$-action.

2.2 Period rings

We will construct the rings $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}$ and $\mathbf{B}_{\mathrm{HT}}$. This will require many intermediary ring constructions, beginning with Fontaine's ring R, which we denote by $\mathcal{O}_{\mathbb{C}_p^\flat}$. ([Col2004, §4] calls it $\widetilde{E}^+$.)

Remark 2.3. I know this learning group is based on [Col2004], but frankly I find it hard to follow. I will instead mainly follow the notation of [Col2019], where the construction of these rings are much more nicely written. I also think it is nice to recognize the construction of Fontaine's R/Colmez's old $\widetilde{E}^+$ as one of the simplest examples of tilting.

We begin with

$$\mathcal{O}_{\mathbb{C}_p^\flat} := \left\{(a_i)_{i\in\mathbb{N}} : a_i \in \mathcal{O}_{\mathbb{C}_p}/p, a_{i+1}^p = a_i\right\}.$$

This is a ring, so we will describe its addition and multiplication laws. For any $a = (a_i)_i \in \mathcal{O}_{\mathbb{C}_p^\flat}$, we can lift each a_i to $\widehat{a}_i \in \mathcal{O}_{\mathbb{C}_p}$, then define

$$a_n^\sharp := \lim_{m\to\infty} \widehat{a}_{n+m}^{p^m} \in \mathcal{O}_{\mathbb{C}_p}.$$

Note that $a_n^\sharp$ is independent of the choice of lifts $\widehat{a}_i$. This procedure $(a_i)_i \mapsto (a_i^\sharp)_i$ in fact defines a bijection of sets between $\mathcal{O}_{\mathbb{C}_p^\flat}$ and sequences $(y_i)_i$ of elements in $\mathcal{O}_{\mathbb{C}_p}$ such that $y_{i+1}^p = y_i$ for all $i \in \mathbb{N}$. Then, for any $a = (a_i)_i$, $b = (b_j)_j$ in $\mathcal{O}_{\mathbb{C}_p^\flat}$, we can see a, b by their lifts to characteristic 0 and define addition and multiplication uniquely by

$$(a+b)_n^\sharp = \lim_{m\to\infty} (a_{n+m}^\sharp + b_{n+m}^\sharp)^{p^m}, \quad (ab)_n^\sharp = a_n^\sharp b_n^\sharp.$$

Observe from the addition rule that $\mathcal{O}_{\mathbb{C}_p^\flat}$ is of characteristic p, and indeed it is routine to check that the Frobenius map $a \mapsto a^p$ is a ring endomorphism of $\mathcal{O}_{\mathbb{C}_p^\flat}$. It is furthermore a complete valuation ring for the valuation $v^\flat(a) := v_p(a_0^\sharp)$, and if $v^\flat(a) > 0$, then $\mathbb{C}_p^\flat := \mathrm{Frac}(\mathcal{O}_{\mathbb{C}_p^\flat}) = \mathcal{O}_{\mathbb{C}_p^\flat}[1/a]$. The $G_{\mathbb{Q}_p}$-action defined component-wise on $\mathcal{O}_{\mathbb{C}_p^\flat}$ extends naturally to $\mathbb{C}_p^\flat$. We state without proof the following two facts of $\mathbb{C}_p^\flat$:

1. $\mathbb{C}_p^\flat$ is an algebraically closed field of characteristic p, complete with respect to the valuation $v^\flat$ defined above.

2. The $G_{\mathbb{Q}_p}$-action on $\mathbb{C}_p^\flat$ induced from that on $\mathcal{O}_{\mathbb{C}_p^\flat}$ is continuous.

Now we define the "one ring to rule them all" $\mathbf{A}_{\mathrm{inf}}$, which is simply the Witt vector ring of $\mathcal{O}_{\mathbb{C}_p^\flat}$. Recall that the Witt vector ring $W(R)$ of some ring R is the set $R^{\mathbb{N}}$ where addition and multiplication are defined component-wise via the *ghost coordinates*

$$W_n(x) = x^{(n)} := \sum_{r=0}^{n} p^r x_r^{p^{n-r}}, \quad x = (x_i)_{i\in\mathbb{N}}.$$

Example 2.4. This construction is good for lifting rings in characteristic p to characteristic 0. The classic example is $W(\mathbb{F}_p) = \mathbb{Z}_p$, or more generally $W(\mathbb{F}_q) = \mathcal{O}_K$ where $K = \mathbb{Q}_p(\mu_{q-1})$ is the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$.

From its construction, we obtain for free that $\mathbf{A}_{\text{inf}}$ is complete with respect to the p-adic topology and $\mathbf{A}_{\text{inf}}/p \simeq \mathcal{O}_{\mathbb{C}_p^\flat}$.

By definition, an element of $\mathbf{A}_{\text{inf}} \coloneqq W(\mathcal{O}_{\mathbb{C}_p^\flat})$ can be written uniquely as $\sum_{n\geq 0}[x_n]p^n$ for some $x_n \in \mathcal{O}_{\mathbb{C}_p^\flat}$ where $[x] = (x, 0, 0, \dots)$ denotes the Teichmüller lift.

Proposition 2.5. *The set map*

$$\begin{aligned} \theta : \mathbf{A}_{\text{inf}} &\to \mathcal{O}_{\mathbb{C}_p} \\ \sum_{n\geq 0}[x_n]p^n &\mapsto \sum_{n\geq 0} x_{n,0}^\sharp p^n. \end{aligned}$$

is a $G_{\mathbb{Q}_p}$-equivariant surjective ring homomorphism.

Proof. The Galois equivariance is obvious since the Galois action happens componentwise. Multiplication also behaves as expected everywhere, so the only difficult part is additivity. It suffices to check additivity of θ mod p^n for all $n \in \mathbb{N}$. (Surjectivity will be clear once we establish what θ looks like at finite levels.)

We will show that $\theta \mod p^n$ is equal to the composition

$$\mathbf{A}_{\text{inf}} \xrightarrow{W(\pi_n)} W(\mathcal{O}_{\mathbb{C}_p}/p) \xrightarrow{\overline{W}_n} \mathcal{O}_{\mathbb{C}_p}/p^n$$

where π_n is the standard projection $\mathcal{O}_{\mathbb{C}_p^\flat} \to \mathcal{O}_{\mathbb{C}_p}/p$ to the n^{th} component, $W(\pi_n)$ is the induced map on the corresponding Witt vector rings (since the Witt ring construction is functorial), W_n is the map on $W(\mathcal{O}_{\mathbb{C}_p}/p^n)$ defining the n^{th} ghost coordinate, and $\overline{W}_n$ is the map such that pre-composition with the mod p projection map is W_n. (Checking that W_n factors through $W(\mathcal{O}_{\mathbb{C}_p}/p)$ is elementary.)

If we view $\mathbf{A}_{\text{inf}}$ as elements of the set $\mathcal{O}_{\mathbb{C}_p^\flat}^{\mathbb{N}}$, then the map θ is given by

$$(x_0, x_1, \dots) \mapsto \sum_{n\geq 0} \left(x_n^{p^{-n}}\right)_0^\sharp \cdot p^n.$$

Thus, for $0 \leq r < n$, it suffices to show

$$\left(x_r^{p^{-r}}\right)_0^\sharp \equiv x_{r,n}^{p^{n-r}} \pmod{p^{n-r}}.$$

But this simply follows from

$$\left(x_r^{p^{-r}}\right)_0^\sharp = \left(x_r^{p^{-r}}\right)_n^{\sharp p^{n-r}} \equiv x_{r,n}^{p^{n-r}} \pmod{p^{n-r}},$$

where the last congruence follows from $x_{r,n}^\sharp \equiv x_{r,n} \pmod{p}$. □

Inverting p on both sides of θ gives us a map (which we also call θ)

$$\theta : \mathbf{A}_{\mathrm{inf}}[1/p] \to \mathcal{O}_{\mathbb{C}_p}[1/p] \simeq \mathbb{C}_p.$$

The source ring is not yet so nice – in particular, it is not a complete DVR – but θ has the convenient property that $\ker\theta$ is principal, generated by the element

$$\xi \coloneqq \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1},$$

where $[\varepsilon]$ is the Teichmüller lift of some $\varepsilon = (\varepsilon_n)_n \in \mathcal{O}_{\mathbb{C}_p^\flat}$ where $\varepsilon_0^\sharp = 1 \neq \varepsilon_1^\sharp$. For a proof where ξ is presented slightly different, see [BC2009, Proposition 4.4.3].

Now we can construct a complete DVR with residue field $\mathbb{C}_p$ by completing $\mathbf{A}_{\mathrm{inf}}[1/p]$ with respect to (ξ). Explicitly, define

$$\mathbf{B}_{\mathrm{dR}}^+ \coloneqq \varprojlim_n \mathbf{A}_{\mathrm{inf}}\left[\frac{1}{p}\right]/\xi^n.$$

In the previous subsection, we said that the p-adic analogue t of $2\pi i$ should be the log of some compatible p^n-th roots of unity. Note that

$$\theta([\varepsilon]-1) = \varepsilon_0^\sharp - 1 = 0,$$

so $[\varepsilon]-1 \in \ker\theta$. In particular, we can make sense of the logarithm

$$t \coloneqq \log([\varepsilon]) = \sum_{n\geq 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n},$$

and this is an element of B_{dR}^+. Even better, Fontaine [Fon1982] showed that t is a uniformizer of B_{dR}^+. Since $G_{\mathbb{Q}_p}$ acts via the cyclotomic character on each ε_n, we can deduce $\sigma(t) = \chi(\sigma)t$ for any $\sigma \in G_{\mathbb{Q}_p}$. So indeed, this t is the appropriate p-adic analogue of $2\pi i$.

Definition 2.6 ($\mathbf{B}_{\mathrm{dR}}$)**.** We denote

$$\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+\left[\frac{1}{t}\right] = \mathrm{Frac}(\mathbf{B}_{\mathrm{dR}}^+).$$

Since t is a uniformizer of $\mathbf{B}_{\mathrm{dR}}^+$, we have $\mathbf{B}_{\mathrm{dR}}$ is a field with a $G_{\mathbb{Q}_p}$-action and a $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}$-graded decreasing filtration given by $\mathrm{Fil}^r\,\mathbf{B}_{\mathrm{dR}} = (t^r)$.

From $\mathbf{B}_{\mathrm{dR}}$, we can get $\mathbf{B}_{\mathrm{HT}}$ easily by just taking the direct sum of the successive quotients in the filtration of $\mathbf{B}_{\mathrm{dR}}$. Succinctly,

Definition 2.7 ($\mathbf{B}_{\mathrm{HT}}$)**.** $\mathbf{B}_{\mathrm{HT}}$ is the $\mathbb{Z}$-graded $\mathbb{C}_p$-algebra

$$\mathbf{B}_{\mathrm{HT}} \coloneqq \bigoplus_{r\in\mathbb{Z}} \mathrm{Fil}^r\,\mathbf{B}_{\mathrm{dR}}/\,\mathrm{Fil}^{r+1}\,\mathbf{B}_{\mathrm{dR}} \simeq \bigoplus_{r\in\mathbb{Z}} \mathbb{C}_p(r),$$

where the last isomorphism follows from the fact that t has cyclotomic $G_{\mathbb{Q}_p}$-action.

One way to motivate the construction of $\mathbf{B}_{\text{cris}}$ is that it is a subring of $\mathbf{B}_{\text{dR}}$ which comes with a natural Frobenius endomorphism. (This is not the case with the whole of $\mathbf{B}_{\text{dR}}$, since the Frobenius automorphism of $\mathbf{A}_{\text{inf}}[1/p]$ does not preserve (ξ).) We first take $\mathbf{A}^0_{\text{cris}}$ to be the divided power envelope of $\mathbf{A}_{\text{inf}}$, namely the $\mathbf{A}_{\text{inf}}$-subalgebra of $\mathbf{A}_{\text{inf}}[1/p]$ generated by $\xi^n/n!$ for all $n \in \mathbb{N}$. Define $\mathbf{A}_{\text{cris}}$ to be the p-adic completion of $\mathbf{A}^0_{\text{cris}}$.

Proposition 2.8. *We state the following facts about $\mathbf{A}_{\text{cris}}$ without proof. (Working with $\mathbf{A}_{\text{cris}}$, as one may expect, is difficult.)*

1. *The natural map $\mathbf{A}^0_{\text{cris}} \to \mathbf{A}_{\text{cris}}$ is injective.*

2. *$\mathbf{A}_{\text{cris}}$ is an integral domain.*

3. *There exists a unique map $j : \mathbf{A}_{\text{cris}} \to \mathbf{B}^+_{\text{dR}}$ such that the following diagram commutes. In addition, it is continuous, $G_{\mathbb{Q}_p}$-equivariant, and injective.*

$$\begin{array}{ccc} \mathbf{A}^0_{\text{cris}} & \hookrightarrow & \mathbf{A}_{\text{inf}}[1/p] \\ \big\downarrow & & \big\downarrow \\ \mathbf{A}_{\text{cris}} & \xrightarrow{\ j\ } & \mathbf{B}^+_{\text{dR}} \end{array}$$

4. *The uniformizer t of $\mathbf{B}^+_{\text{dR}}$ is also an element of $\mathbf{A}_{\text{cris}}$.*

5. *The Frobenius φ on $\mathbf{A}_{\text{inf}}[1/p]$ induced from the Frobenius on $\mathcal{O}_{\mathbb{C}_p^\flat}$ acts stably on $\mathbf{A}^0_{\text{cris}}$ (and hence on $\mathbf{A}_{\text{cris}}$.)*

6. *The Frobenius φ on $\mathbf{A}_{\text{cris}}$ is injective, and $\varphi(t) = pt$.*

With these facts in hand, we can define the subring $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{dR}}$ with a Frobenius action.

Definition 2.9 ($\mathbf{B}_{\text{cris}}$)**.** Denote $t = \log([\varepsilon]) \in \mathbf{A}_{\text{cris}} \subset \mathbf{B}^+_{\text{dR}}$ as above. Then, we define

$$\mathbf{B}_{\text{cris}} \coloneqq \mathbf{A}_{\text{cris}} \left[\frac{1}{t} \right].$$

It is a subring of $\mathbf{B}_{\text{dR}}$ with a Frobenius endomorphism φ extending that of $\mathbf{A}_{\text{cris}}$, so in particular $\varphi(t) = pt$.

Finally, we define $\mathbf{B}_{\text{st}}$, for which we will be even more terse. (We include its construction merely for completeness.)

Definition 2.10 ($\mathbf{B}_{\text{st}}$)**.** Let $p^\flat \in \mathcal{O}_{\mathbb{C}_p^\flat}$ be a compatible system of p^n-th roots of p, and let $u \in \mathbf{B}_{\text{dR}}$ be the element given by

$$u = \log \frac{[p^\flat]}{p} = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \left(\frac{[p^\flat]}{p} - 1 \right)^r.$$

Define $\mathbf{B}_{\mathrm{st}}$ as

$$\mathbf{B}_{\mathrm{st}} \coloneqq \mathbf{B}_{\mathrm{cris}}[u].$$

Like with $\mathbf{B}_{\mathrm{cris}}$, we tabulate some facts without proof.

Proposition 2.11. $\mathbf{B}_{\mathrm{st}}$ *inherits from* $\mathbf{B}_{\mathrm{dR}}$ *a Galois action, and we extend the Frobenius* φ *on* $\mathbf{B}_{\mathrm{cris}}$ *to* $\mathbf{B}_{\mathrm{st}}$ *by asserting* $\varphi(u) = pu$. *We can also endow* $\mathbf{B}_{\mathrm{st}}$ *with a monodromy operator* $N = -\frac{d}{du}$, *with* $\mathbf{B}_{\mathrm{cris}} = \ker N$. *The Galois action commutes with both* φ *and* N, *and the latter two satisfy the relation*

$$N\varphi = p\varphi N.$$

We conclude this subsection with Fontaine's "fundamental exact sequence" which realizes $\mathbb{Q}_p$ inside $\mathbf{B}_{\mathrm{cris}}$ (and hence in $\mathbf{B}_{\mathrm{st}}$):

$$0 \to \mathbb{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\operatorname{Fil}^0 \mathbf{B}_{\mathrm{dR}} \to 0. \tag{1}$$

2.3 Classification of p-adic representations

If number theory is about studying the group $G_{\mathbb{Q}}$, and groups are studied through their representation theory, then it makes sense for us to consider representations of $G_{\mathbb{Q}}$. Even more, L-functions and the general Langlands philosophy suggests that we can study representations of global Galois groups via those of local Galois groups.

Let $K/\mathbb{Q}_p$ be a finite extension and G_K its absolute Galois group. We denote $\mathsf{Rep}_F(G_K)$ as the category of all F-representations of G_K for any field F. As geometry naturally gives us representations over local fields via cohomology, we will take $F = \mathbb{Q}_\ell$ for some prime ℓ.

When $\ell \neq p$, the incompatibility of the topologies[2] of G_K and $\mathrm{GL}_n(\mathbb{Q}_\ell)$ forces the study of ℓ-adic representations of G_K to be quite nice. By "nice" we mean they are all potentially semistable, which we first define.

Definition 2.12 (Potentially semistable)**.** Let $K/\mathbb{Q}_p$ be a finite extension and $\ell \neq p$ primes. An ℓ-adic representation ρ of G_K is **semistable** if the inertia subgroup I_K acts via unipotent operators. It is **potentially semistable** if there exists some finite extension K'/K such that $\rho|_{G_{K'}}$ is semistable.

Remark 2.13. In general, the adjective "potentially" for some property P of a G_K-representation ρ indicates that ρ satisfies P after restricting to $G_{K'}$ for some finite K'/K.

Theorem 2.14 (Grothendieck ℓ-adic Monodromy)**.** *Every representation in* $\mathsf{Rep}_{\mathbb{Q}_\ell}(G_K)$ *is potentially semistable.*

[2] To illustrate, note there is no continuous homomorphism $\mathbb{Q}_p \to \mathbb{Q}_\ell$ for $\ell \neq p$.

On the flip side, the topologies of G_K and $\mathrm{GL}_n(\mathbb{Q}_p)$ are more similar, so there is no such immediate nice classification. The study of p-adic representations of G_K is p-adic Hodge theory.

The general shape of such study reflects our motivation for $\mathbf{B}_{\mathrm{dR}}$ in §2.1: after extending coefficients to a large enough ring (the period rings above), representations coming from geometry (e.g., étale cohomology) should decompose into better-understood spaces (e.g., de Rham cohomology). Note an underlying implication here is that we only care about Galois representations coming from geometry. This may seem restrictive, but it is actually not so unreasonable. For instance, there is the Fontaine–Mazur conjecture, which has been proved in some cases.

Conjecture 2.15 (Fontaine–Mazur)**.** *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ be an irreducible representation which is de Rham at p and unramified outside finitely many primes. Then, ρ occurs as the twist of some subquotient of the étale cohomology of a smooth projective variety $X/\mathbb{Q}$.*

Remark 2.16. In the $n = 2$ case, which in fact was the original formulation of the conjecture [FM1997], if we further suppose ρ to be odd, then the conjecture predicts that ρ is the Galois representation associated to a cuspidal eigenform.

We revisit the étale-to-de Rham comparison theorem to motivate the general formalism of Fontaine. For a smooth proper variety over $\mathbb{Q}_p$, we have a $G_{\mathbb{Q}_p}$-equivariant isomorphism

$$H^i_{\text{ét}}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}.$$

From our construction of $\mathbf{B}_{\mathrm{dR}}$ above and the Ax-Sen-Tate Theorem, we have $\mathbf{B}_{\mathrm{dR}}^{G_{\mathbb{Q}_p}} = \mathbb{Q}_p$, so taking Galois invariants on both sides gives

$$\left(H^i_{\text{ét}}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}\right)^{G_{\mathbb{Q}_p}} \xrightarrow{\sim} H^i_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

The right-hand side is a nice linear algebraic object (a filtered finite-dimensional $\mathbb{Q}_p$-vector space). In general, Fontaine's formalism will compare various types of Galois representations to different semilinear algebraic objects.

We state Fontaine's formalism in great generality, but really the case that we care about will be when $G = G_K$, $F = \mathbb{Q}_p$, and B one of the period rings defined above. We take the following setup:

- G is a topological group,
- B is a commutative integral topological ring with a continuous G-action,
- $E \coloneqq B^G \subset G$,
- $F \subset E$ is some closed subfield,

- $\mathsf{Rep}_B(G)$ is the category of B-representations of G, where
  - objects: free B-modules of finite rank with continuous semilinear G-action,
  - morphisms: G-equivariant B-module maps.

We want to send an object in $\mathsf{Rep}_F(G)$ to a nice semilinear object. This is done via the functor

$$D_B : \mathsf{Rep}_F(G) \xrightarrow{B\otimes_F(-)} \mathsf{Rep}_B(G) \xrightarrow{(-)^G} \mathsf{Mod}_E,$$

i.e., $D_B(V) \coloneqq (B \otimes_F V)^G$. This space comes equipped with a natural map

$$\alpha_V : B \otimes_E D_B(V) \to B \otimes_F V$$
$$b \otimes \delta \mapsto b\delta,$$

which in some sense recovers V from $D_B(V)$.

Definition 2.17 (Regular)**.** We say B is (F, G)-regular if (1) $E = B^G = \mathrm{Frac}(B)^G$ and (2) if $0 \neq b \in B$ and $Fb \subset B$ is G-stable, then $b \in B^\times$.

Definition 2.18. Suppose B is (F, G)-regular, in which case α_V is injective for all $V \in \mathsf{Rep}_F G$. We say such a V is a **B-admissible** representation if it satisfies any one of the following equivalent conditions:

1. $\dim_E D_B(V) = \dim_F(V)$
2. α_V is an isomorphism
3. $B \otimes_F V \simeq B^{\oplus r}$ in $\mathsf{Rep}_B G$ for some $r \in \mathbb{N}$.

We denote $\mathsf{Rep}_F^B G \subset \mathsf{Rep}_F G$ as the full subcategory of B-admissible G-representations.

To ensure that this category is nice and worth talking about:

Proposition 2.19. *The (sub)category $\mathsf{Rep}_F^B(G)$ is Tannakian, and the functor D_B : $\mathsf{Rep}_F^B(G) \to \mathsf{Vect}_E$ is exact, faithful, and commutes with tensors.*

Example 2.20. Take $G = G_{\mathbb{Q}_p}$, $B = \mathbf{B}_{\mathrm{dR}}$, and $F = E = B^G = \mathbb{Q}_p$. We check that $\mathbf{B}_{\mathrm{dR}}$ is $(\mathbb{Q}_p, G_{\mathbb{Q}_p})$-regular: (1) we just stated $E = B^G$, and (2) any nonzero $b \in \mathbf{B}_{\mathrm{dR}}$ (regardless of the G-stable line condition) is a unit since $\mathbf{B}_{\mathrm{dR}}$ is a field.

As stated before, we will focus on when $G = G_K$, $F = \mathbb{Q}_p$, and B is one of the period rings defined in the previous subsection. For any $\tau \in \{\mathrm{HT}, \mathrm{dR}, \mathrm{cris}, \mathrm{st}\}$, it is known that $\mathbf{B}_\tau$ is $(\mathbb{Q}_p, G_K)$-regular.

For each τ above, our functor $D_\tau \coloneqq D_{\mathbf{B}_\tau}$ is from $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ to Vect_{E_τ}, where $E_\tau \coloneqq (\mathbf{B}_\tau)^{G_K}$. Restrict D_τ to $\mathsf{Rep}_{\mathbb{Q}_p}^\tau(G_K) \coloneqq \mathsf{Rep}_{\mathbb{Q}_p}^{B_\tau}(G_K)$. Fontaine's program identifies a subcategory $\mathcal{C}_\tau \subset \mathsf{Vect}_{E_\tau}$ such that the functor $D_\tau : \mathsf{Rep}_{\mathbb{Q}_p}^\tau(G_K) \to \mathcal{C}_\tau$ is an equivalence of categories. We describe these $\mathcal{C}_\tau$:

$\tau = \text{HT}$: $\mathcal{C}_{\text{HT}} = \mathsf{Grad}_K^{\mathbb{Z}}$, objects being finite-dimensional $\mathbb{Z}$-graded K-vector spaces $V \simeq \bigoplus_{n\in\mathbb{Z}} V^{(n)}$.

$\tau = \text{dR}$: $\mathcal{C}_{\text{dR}} = \mathsf{Fil}_K$, objects being $\mathbb{Z}$-graded decreasing separated exhaustive filtrations of finite-dimensional K-vector spaces.

$\tau = \text{cris}$: $\mathcal{C}_{\text{cris}} = \mathsf{FilM}_K^{\varphi}$, objects being the data $(V, \varphi, \operatorname{Fil}^\bullet V_K)$, where

(a) V is a finite-dimensional K_0-vector space, where $K_0 = K \cap \mathbb{Q}_p^{\text{ur}}$

(b) $\varphi : V \xrightarrow{\sim} V$ is a σ-semilinear automorphism, where $\sigma : K_0 \to K_0$ is a lift of the Frobenius

(c) $\operatorname{Fil}^\bullet V_K \in \operatorname{Fil}_K$ is a filtration of $V_K \coloneqq V \otimes_{K_0} K$.

$\tau = \text{st}$: $\mathcal{C}_{\text{st}} = \mathsf{FilM}_K^{(\varphi,N)}$, objects being the data $(V, \varphi, N, \operatorname{Fil}^\bullet V_K)$, where

(a) $(V, \varphi, \operatorname{Fil}^\bullet V_K) \in \mathsf{FilM}_K^{\varphi}$

(b) $N : V \to V$ is a K_0-linear nilpotent endomorphism such that $N\varphi = p\varphi N$.

Note that $\mathcal{C}_{\text{cris}} \subset \mathcal{C}_{\text{st}}$ (take $N = 0$) and $\mathcal{C}_{\text{st}} \subset \mathcal{C}_{\text{dR}}$ (only remember $\operatorname{Fil}^\bullet V_K$). Equivalently, $\mathsf{Rep}_{\mathbb{Q}_p}^{\text{cris}} G_K \subset \mathsf{Rep}_{\mathbb{Q}_p}^{\text{st}} G_K \subset \mathsf{Rep}_{\mathbb{Q}_p}^{\text{dR}} G_K$, which makes sense from the inclusions $\mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\text{st}} \subset \mathbf{B}_{\text{dR}}$. We also have the inclusion $\mathsf{Rep}_{\mathbb{Q}_p}^{\text{pst}} G_K \subset \mathsf{Rep}_{\mathbb{Q}_p}^{\text{dR}} G_K$, where pst indicates potentially semistable. By [Ber2002], the reverse inclusion is also true.

Theorem 2.21 (p-adic monodromy)**.** *The inclusion* $\mathsf{Rep}_{\mathbb{Q}_p}^{\text{pst}} G_K \subset \mathsf{Rep}_{\mathbb{Q}_p}^{\text{dR}} G_K$ *is an equivalence of categories.*

2.4 (φ, Γ)-modules

The above classification is nice, but suppose we wanted to detach a bit from the geometric source of Galois representations and consider all Galois representations. Without any extra conditions, is there an equivalence of categories between $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ and some nice category, perhaps of φ-semilinear objects?

The development of (φ, Γ)-modules can be seen through the following two remarkable facts:

1. Let K be a p-adic field and K_∞/K an infinitely ramified algebraic extension such that the Galois group of its Galois closure is a p-adic Lie group. Then,

   (a) The theory of norm fields gives a functorial equivalence between the category of separable algebraic extensions of K_∞ and that of a characteristic p local field E. Non-canonically, we have $E \simeq k_\infty(\!(t)\!)$, where k_∞ is the residue field of K_∞.

   (b) There is a canonical topological isomorphism

$$\operatorname{Gal}(K_\infty^{\text{sep}}/K_\infty) \simeq \operatorname{Gal}(E^{\text{sep}}/E).$$

2. There is an equivalence of categories between $\mathsf{Rep}_{\mathbb{Q}_p}(G_E)$ and a certain category of **étale φ-modules**, whose objects have a φ-semilinear action.

So, in loose terms, a category of semilinear algebraic nature which is equivalent to $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ must have (a) a semilinear φ-action coming from $\mathrm{Gal}(K_\infty^{\mathrm{sep}}/K_\infty)$ and (b) a $\Gamma = \mathrm{Gal}(K_\infty/K)$-action. This is the essence of (φ, Γ)-modules. We proceed directly to the setup:

- Let $K/\mathbb{Q}_p$ be a finite extension and $K_\infty = K(\varepsilon_n^\sharp)_n$. Let E be the corresponding characteristic p local field given by the theory of norm fields.
- Let $\chi : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character and denote the kernel by H_K. Denote $E = E_K^{\mathrm{sep}}$; we have $\mathrm{Gal}(E/E_K) = H_K$.
- Denote $\Gamma_K := \mathrm{Gal}(K_\infty/K) = G_K/H_K$.
- Denote $\widetilde{\mathbf{A}} = W(\mathbb{C}_p^\flat)$ and define the element $\pi = [\varepsilon] - 1 \in \widetilde{\mathbf{A}}$. It has φ-action given by
$$\varphi(\pi) = [\varepsilon]^p - 1 = (1+\pi)^p - 1.$$
The $G_{\mathbb{Q}_p}$-action on π factors through $\Gamma_{\mathbb{Q}_p}$.
- Denote $\mathbf{A}_{\mathbb{Q}_p}$ as the closure of $\mathbb{Z}_p[\pi, \pi^{-1}]$ in $\widetilde{\mathbf{A}}$, or in other words,
$$\mathbf{A}_{\mathbb{Q}_p} := \left\{ \sum_{r \in \mathbb{Z}} a_r \pi^r : a_r \in \mathbb{Z}_p, a_r \to 0 \text{ as } r \to -\infty \right\}.$$
- Define $\mathbf{B}_{\mathbb{Q}_p} := \mathbf{A}_{\mathbb{Q}_p}[1/p]$ and $\widetilde{\mathbf{B}} := \widetilde{\mathbf{A}}[1/p]$. Denote $\mathbf{B}$ as the closure of the maximal unramified extension of $\mathbf{B}_{\mathbb{Q}_p}$ in $\widetilde{\mathbf{B}}$, and define $\mathbf{A} := \mathbf{B} \cap \widetilde{\mathbf{A}}$. By Ax-Sen-Tate, we have $\mathbf{B}^{\varphi=1} = \mathbb{Q}_p$ and $\mathbf{B}^{H_K} = \mathbf{B}_K$.

Definition 2.22. A **(φ, Γ)-module** D over $\mathbf{B}_K$ is a finite-dimensional $\mathbf{B}_K$-vector space equipped with semi-linear actions of φ and Γ_K which commute with each other. An **étale** (φ, Γ)-module D over $\mathbf{B}_K$ is one where the linearization $\varphi^*(D) \to D$ is an isomorphism.

Theorem 2.23. *There is an equivalence of categories between $\mathsf{Rep}_{\mathbb{Q}_p}(G_K)$ and the category of étale (φ, Γ)-modules over $\mathbf{B}_K$, where the functors D and V are given by*
$$D(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_K}$$
$$V(D) = (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}.$$

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