EICHLER-SHIMURA ISOMORPHISM

HAHN LHEEM

Contents

1. Set-Up

Throughout, we fix a congruence subgroup Γ, meaning a subgroup of $\mathrm{SL}_2(\mathbb{Q})$ such that $(\Gamma : \Gamma(N))$ has finite index for some $N \in \mathbb{Z}_{\geq 0}$. We denote $Y_\Gamma = \Gamma\backslash\mathbb{H}$ and $X_\Gamma = \Gamma\backslash\mathbb{H}^*$, where $\mathbb{H}^* := \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$.

1.1. Modular Forms.

Fix a weight $k \in \mathbb{Z}_{>0}$. (In fact, the Eichler-Shimura isomorphism will require $k \geq 2$.) We will define the space of modular forms (resp., cusp forms) of weight k with respect to the congruence subgroup Γ.

Modular forms with respect to Γ are holomorphic functions on $\mathbb{H}$ obeying a certain invariance ("automorphy") property, which we now describe. For each weight $k \in \mathbb{Z}_{>0}$, the space of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ admits a right $\mathrm{GL}_2^+(\mathbb{Q})$-action: given such an f and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$, define the action by

$$f|_k\alpha(z) := (\det \alpha)^{k/2} j(\alpha, z)^{-k} f(\alpha z),$$

where $j(\alpha, z) \coloneqq cz + d$ and αz is the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{H}$ by linear fractional transformations. A modular form of weight k with respect to Γ must satisfy the *automorphy equation* $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

Modular forms must also be "holomorphic" at the cusps. Let $s \in \mathbb{P}^1(\mathbb{Q})$ and choose some $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that $\gamma(i\infty) = s$. For any $z \in \mathbb{H}$, denote Γ_z as the stabilizer of z in Γ. We then have

$$\begin{aligned} \gamma^{-1}\Gamma_s\gamma \cdot \{\pm I_2\} &= (\gamma^{-1}\Gamma\gamma)_{i\infty} \cdot \{\pm I_2\} \\ &\subseteq \mathrm{SL}_2(\mathbb{Z})_{i\infty} = \pm \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \\ \implies \gamma^{-1}\Gamma_s\gamma \cdot \{\pm I_2\} &= \pm \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix} \end{aligned}$$

for some $h > 0$. In the case where k is odd, we say that the cusp s is **regular** if $\gamma^{-1}\Gamma_s\gamma$ is generated by $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$, and we call it **irregular** otherwise. This distinction will be important in computing the dimension of the space of modular forms.
Let $\beta \in \Gamma_s$ such that $\gamma^{-1}\beta\gamma = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$. On the one hand, we have by definition

$$(f|_k\gamma)|_k\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)(z) = f|_k\gamma(z+h).$$

On the other hand, noting $\beta \in \Gamma$, we have

$$(f|_k\gamma)|_k\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) = (f|_k\gamma)|_k\gamma^{-1}\beta\gamma = f|_k\gamma,$$

hence $f|_k\gamma(z+h) = f|_k\gamma(z)$. This means we have a Fourier expansion of $f|_k\gamma$ around $z = s$ in the form

$$f|_k\gamma(z) = \sum_{n=-\infty}^{\infty} c_n(f,s) e^{\frac{2\pi i n z}{h}}.$$

Definition 1.1 (Modular and Cusp Forms)**.** Let $k \in \mathbb{Z}_{>0}$ and Γ a congruence subgroup. A **modular form** of weight k with respect to Γ is a holomorphic $f : \mathbb{H} \to \mathbb{C}$ such that

(1) It satisfies the automorphy equation for Γ, namely $f|_k\gamma = f$ for all $\gamma \in \Gamma$,
(2) For each cusp s of Γ and $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that $\gamma(i\infty) = s$, the Fourier expansion of $f|_k\gamma$ at $z = s$ as given above satisfies $c_n(f,s) = 0$ for all $n < 0$.

We say a modular form (of weight k with respect to Γ) is a **cusp form** (with the same adjectives) if $c_n(f,s) = 0$ for all $n \leq 0$ in the Fourier expansion at the cusps.

We denote $\mathcal{M}_k(\Gamma)$ (resp., $\mathcal{S}_k(\Gamma)$) as the space of modular (resp., cusp) forms of weight k with respect to Γ. Note that both carry a natural $\mathbb{C}$-vector space structure, although for the Eichler-Shimura isomorphism, we will view them as $\mathbb{R}$-vector spaces.

One important feature of $\mathcal{S}_k(\Gamma)$ is that it has an inner product, called the **Petersson inner product**. More generally, for $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{M}_k(\Gamma)$, we define

$$\langle f, g \rangle_\Gamma \coloneqq \int_{Y_\Gamma} f(z)\overline{g(z)}\, y^k \frac{dx\,dy}{y^2},$$

where $z = x + iy$. It is evident that $\langle \cdot, \cdot \rangle_\Gamma$ is a non-degenerate $\mathbb{R}$-bilinear form, and positive-definite Hermitian as an inner product on $\mathcal{S}_k(\Gamma)$.

1.2. **Interior Cohomology.** Let L be a discrete left Γ-module, and define $\underline{L}_\Gamma$ as the sheaf of continuous sections of the projection $\Gamma\backslash(\mathbb{H}\times L) \twoheadrightarrow \Gamma\backslash\mathbb{H}$. We will now describe the exact sequence

$$0 \to H^1_!(Y_\Gamma, \underline{L}_\Gamma) \to H^1(Y_\Gamma, \underline{L}_\Gamma) \to H^1_\partial(Y_\Gamma, \underline{L}_\Gamma)$$

between interior cohomology, the usual sheaf cohomology, and boundary cohomology.

Let $\mathcal{F} \in \mathrm{Sh}(Y_\Gamma)$. Let $\{c_i\}$ be the cusps of X_Γ, let $D_{i,\varepsilon} \subset Y_\Gamma$ be a punctured open neighborhood around c_i homeomorphic to the punctured open disk $D(0,\varepsilon)^*$, and denote $D_\varepsilon \coloneqq \bigsqcup_i D_{i,\varepsilon}$ Denote $K_\varepsilon = Y_\Gamma \setminus D_\varepsilon$, and let $\mathcal{F}_{K_\varepsilon}$ be the sheaf on Y_Γ whose sections have support contained in K_ε. If $j_\varepsilon : D_\varepsilon \hookrightarrow Y_\Gamma$ is the natural inclusion, then we have the short exact sequence $0 \to \mathcal{F}_{K_\varepsilon} \to \mathcal{F} \to j_{\varepsilon,!}j_\varepsilon^*\mathcal{F} \to 0$. Denoting $H^i_{K_\varepsilon}(Y_\Gamma,\mathcal{F}) \coloneqq H^i(Y_\Gamma, \mathcal{F}_{K_\varepsilon})$, we have the long exact sequence

$$\cdots \to H^{i-1}(D_\varepsilon,\mathcal{F}) \to H^i_{K_\varepsilon}(Y_\Gamma,\mathcal{F}) \to H^i(Y_\Gamma,\mathcal{F}) \to H^i(D_\varepsilon,\mathcal{F}) \to H^{i+1}_{K_\varepsilon}(Y_\Gamma,\mathcal{F}) \to \cdots.$$

We now define the **compactly supported cohomology** and **boundary cohomology** to be

$$H^i_c(Y_\Gamma,\mathcal{F}) \coloneqq \varinjlim H^i_{K_\varepsilon}(Y_\Gamma,\mathcal{F})$$
$$H^i_\partial(Y_\Gamma,\mathcal{F}) \coloneqq \varinjlim H^1(D_\varepsilon,\mathcal{F}),$$

where the directed system for both consists of maps in the direction of decreasing ε. Taking the direct limit of the long exact sequence above with $\mathcal{F} = \underline{L}_\Gamma$, and noting that direct limits preserve exactness, we get the exact sequence

$$H^0_\partial(Y_\Gamma, \underline{L}_\Gamma) \to H^1_c(Y_\Gamma, \underline{L}_\Gamma) \to H^1(Y_\Gamma, \underline{L}_\Gamma) \to H^1_\partial(Y_\Gamma, \underline{L}_\Gamma).$$

Finally, we define the **interior cohomology**, denoted $H^1_!(Y_\Gamma, \underline{L}_\Gamma)$, as the image of the map $H^1_c(Y_\Gamma, \underline{L}_\Gamma) \to H^1(Y_\Gamma, \underline{L}_\Gamma)$. We then have the desired short exact sequence

$$0 \to H^1_!(Y_\Gamma, \underline{L}_\Gamma) \to H^1(Y_\Gamma, \underline{L}_\Gamma) \to H^1_\partial(Y_\Gamma, \underline{L}_\Gamma).$$

2. Eichler-Shimura Map

We now construct the Eichler-Shimura map between modular forms of weight k with respect to Γ and a certain first cohomology group of Y_Γ. Afterwards, we will prove that this map is in fact an isomorphism of $\mathbb{R}$-vector spaces. (The map being Hecke-equivariant was proven in class.)

Let $n = k-2$. For the cohomology side of the Eichler-Shimura map, we will set $L = L(n,\mathbb{R})$ to be $\mathbb{R}[X,Y]_n$, the space of homogeneous polynomials in $\mathbb{R}[X,Y]$ of degree n. Upon specifying an $\mathbb{R}$-basis $\{e_1,e_2\}$ of $\mathbb{R}^2$, we have an $\mathbb{R}$-vector space isomorphism $L(n,\mathbb{R}) \simeq (\mathrm{Sym}^n\,\mathbb{R}^2)^\vee$ where $X^iY^{n-i} \mapsto e_1^{\vee i} \otimes e_2^{\vee(n-i)}$. Note that $\mathrm{Sym}^n\,\mathbb{R}^2$, seen as an $\mathbb{R}$-subspace of $(\mathbb{R}^2)^{\otimes n}$, has a natural $\mathrm{SL}_2(\mathbb{R})$-action induced from the standard representation (i.e., left-multiplication) on $\mathbb{R}^2$. Passing this $\mathrm{SL}_2(\mathbb{R})$-action over the specified isomorphism above gives an $\mathrm{SL}_2(\mathbb{R})$-module, hence a Γ-module, structure for $L(n,\mathbb{R})$.

Since $L = L(n,\mathbb{R})$ is flat over $\mathbb{R}$, tensoring $\underline{L}_\Gamma$ over $\underline{\mathbb{R}}$ with the holomorphic de Rham complex

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_{Y_\Gamma} \to \Omega^1_{Y_\Gamma} \to 0$$

yields the short exact sequence

$$0 \to \underline{L(n,\mathbb{C})}_\Gamma \to \underline{L(n,\mathbb{R})}_\Gamma \otimes_{\underline{\mathbb{R}}} \mathcal{O}_{Y_\Gamma} \to \underline{L(n,\mathbb{R})}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_{Y_\Gamma} \to 0,$$

where $L(n,\mathbb{C}) = \mathbb{C}[X,Y]_n$ and $\underline{\mathbb{R}}$ is the constant sheaf for $\mathbb{R}$ on Y_Γ. Note we have a natural map

$$\begin{aligned} \mathcal{M}_k(\Gamma) &\to H^0(Y_\Gamma, \underline{L(n,\mathbb{R})}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_{Y_\Gamma}) \\ f &\mapsto \omega_f(z) := (X - zY)^n \otimes f(z)\, dz, \end{aligned} \tag{1}$$

where, given a basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ and the corresponding basis $\{e_1^i \otimes e_2^{n-i} : 0 \le i \le n\}$ for $\mathbb{R}^{n+1}$, the term $(X - zY)^n$ is the product

$$(X^n, X^{n-1}Y, \dots, Y^n) \begin{pmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & (-1)^{n-1}\binom{n}{1} & & \\ (-1)^n & & & \end{pmatrix} \begin{pmatrix} z^n \\ z^{n-1} \\ \vdots \\ 1 \end{pmatrix}.$$

We also have the coboundary map from the de Rham complex (post-tensoring)

$$H^0(Y_\Gamma, \underline{L(n,\mathbb{R})}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_{Y_\Gamma}) \xrightarrow{\delta} H^1(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma). \tag{2}$$

Composing the maps from (1) and (2), we get a natural map

$$\begin{aligned} \mathcal{M}_k(\Gamma) &\to H^1(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma) \\ f(z) &\mapsto [\omega_f] := \delta(\omega_f). \end{aligned}$$

Denoting $\overline{\mathcal{M}_k(\Gamma)} = \{\overline{f(z)} : f \in \mathcal{M}_k(\Gamma)\}$, the above map naturally extends to the **Eichler-Shimura map**

$$\mathrm{ES} : \mathcal{S}_k(\Gamma) \oplus \overline{\mathcal{M}_k(\Gamma)} \to H^1(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma).$$

We examine more closely the image of cusp forms under this map. Because $f \in \mathcal{S}_k(\Gamma)$ vanishes at the cusps, so does ω_f, hence we have $\omega_f \in H^0_c(Y_\Gamma, \underline{L(n,\mathbb{R})}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_{Y_\Gamma})$.[1] Since sheaf cohomology commutes with direct limits [Sta25, Tag 0739], we have $[\omega_f] \in H^1_c(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma)$, which maps into $H^1_!(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma) \subset H^1(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma)$. Thus, the Eichler-Shimura map restricts to

$$\mathrm{ES} : \mathcal{S}_k(\Gamma) \oplus \overline{\mathcal{S}_k(\Gamma)} \to H^1_!(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma).$$

The objective of the proceeding two sections is to prove the following theorem.

Theorem 2.1 (Eichler-Shimura Isomorphism)**.** *The Eichler-Shimura map*

$$\mathrm{ES} : \mathcal{S}_k(\Gamma) \oplus \overline{\mathcal{M}_k(\Gamma)} \to H^1(Y_\Gamma, \underline{L(n,\mathbb{C})}_\Gamma)$$

[1] This justification is incomplete. See the end of §5.5 for a complete explanation.

is an isomorphism of $\mathbb{R}$*-vector spaces, and it restricts to an isomorphism*

$$\mathrm{ES} : \mathcal{S}_k(\Gamma) \oplus \overline{\mathcal{S}_k(\Gamma)} \to H^1_!(Y_\Gamma, \underline{L(n, \mathbb{C})}_\Gamma).$$

Furthermore, these two isomorphisms are Hecke-equivariant.

Remark 2.2. The Hecke-equivariance was proved in class, so we will omit its proof in this exposition.

3. Proof of Isomorphism I: Injectivity

Simply put, we seek to prove ES is both injective and surjective. For injectivity, we primarily follow the argument in [Shi71, §8.2] (likewise [Hid93, §6.2]). For surjectivity, we provide two proofs, the first originating from [Shi71], and both presented in [Hid93].

Our method of proof will be to construct a pairing $A : \mathcal{S}_k(\Gamma) \times \mathcal{M}_k(\Gamma) \to \mathbb{C}$ which factors through the Eichler-Shimura map. The result will follow from the fact that A is non-degenerate, which we will show by expressing A in terms of the Petersson inner product.

The construction of such a pairing begins on the side of cohomology, where for a general Γ-module L we have a cup product [Hat02, p.209]

$$H^1_c(Y_\Gamma, \underline{L}_\Gamma) \times H^1(Y_\Gamma, \underline{L}_\Gamma) \xrightarrow{\cup} H^2_c(Y_\Gamma, \underline{L}_\Gamma \otimes \underline{L}_\Gamma).$$

Note that on differential forms, the cup product in de Rham cohomology is simply the wedge product.

In our setting of interest $L = L(n, \mathbb{C})$, we seek $\mathbb{C}$-linear maps $H^2_c(Y_\Gamma, \underline{L}_\Gamma \otimes \underline{L}_\Gamma) \xrightarrow{B} H^2_c(Y_\Gamma, \underline{\mathbb{C}}) \xrightarrow{\mathrm{Tr}} \mathbb{C}$. With such maps, we can then define a pairing A on $\mathcal{S}_k(\Gamma) \times \mathcal{M}_k(\Gamma)$ by the following composition.

$$\begin{aligned} A : \mathcal{S}_k(\Gamma) \times \mathcal{M}_k(\Gamma) &\xrightarrow{\mathrm{id} \times \overline{(\cdot)}} \mathcal{S}_k(\Gamma) \times \overline{\mathcal{M}_k(\Gamma)} \\ &\xrightarrow{\mathrm{ES}} H^1_c(Y_\Gamma, \underline{L(n, \mathbb{C})}_\Gamma) \times H^1(Y_\Gamma, \underline{L(n, \mathbb{C})}_\Gamma) \\ &\xrightarrow{\cup} H^2_c(Y_\Gamma, \underline{L(n, \mathbb{C})}_\Gamma \otimes \underline{L(n, \mathbb{C})}_\Gamma) \\ &\xrightarrow{B} H^2_c(Y_\Gamma, \underline{\mathbb{C}}) \\ &\xrightarrow{\mathrm{Tr}} \mathbb{C}. \end{aligned}$$

We first define the map B; it suffices to construct a $\mathbb{C}$-bilinear map (which, abusing notation, we also call B)

$$B : L(n, \mathbb{C}) \otimes L(n, \mathbb{C}) \to \mathbb{C}.$$

In the case $n = 1$, we have a very natural choice for B given by the determinant

$$\begin{aligned} L(1, \mathbb{C}) \otimes L(1, \mathbb{C}) &\xrightarrow{\det} \mathbb{C} \\ (aX + bY, cX + dY) &\mapsto \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{aligned}$$

Note we can write the determinant as

$$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = (a\ b)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}.$$

Specify an isomorphism $L(1,\mathbb{C}) \simeq (\mathbb{C}^2)^\vee$ (i.e., choose a basis $\{e_1, e_2\}$ of $\mathbb{C}^2$), and let Θ_1 be the linear operator on $\mathbb{C}^2$ with matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with respect to $\{e_1, e_2\}$. Let Θ_n be the linear operator on $\operatorname{Sym}^n \mathbb{C}^2$ induced from Θ_1, meaning it has matrix representation with respect to $\{e_1^i \otimes e_2^{n-i}\}$ as

$$\Theta_n := \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & (-1)^{n-1}\binom{n}{1} & & \\ (-1)^n & & & \end{pmatrix}.$$

Identifying $L(n,\mathbb{C}) \simeq (\operatorname{Sym}^n \mathbb{C}^2)^\vee$ means Θ_n induces the linear operator $\Theta_n^{-\intercal}$ on $L(n,\mathbb{C})$. We now define our map $B : L(n,\mathbb{C})^{\otimes 2} \to \mathbb{C}$ as

$$B\left(\sum_{i=0}^n a_i X^i Y^{n-i} \otimes \sum_{j=0}^n b_j X^j Y^{n-j}\right) = (a_i)_i^\intercal \Theta_n^{-\intercal} (b_j)_j = \sum_{k=0}^n a_{n-k} b_k (-1)^{n-k} \binom{n}{k}^{-1}.$$

Remark 3.1. Because this computation will be used later, we note here that

$$\begin{aligned} B\left((X - zY)^n \otimes (X - \overline{z}Y)^n\right) &= \sum_{k=0}^n (-1)^k \binom{n}{k} z^k (-1)^{n-k} \binom{n}{n-k} \overline{z}^{n-k} (-1)^{n-k} \binom{n}{k}^{-1} \\ &= \sum_{k=0}^n (-1)^k z^k \overline{z}^{n-k} \binom{n}{k} \\ &= (\overline{z} - z)^n = (-2iy)^n. \end{aligned}$$

The trace map is simply given by the isomorphism

$$H_c^2(Y_\Gamma, \underline{\mathbb{C}}) \simeq H_c^2(Y_\Gamma; \mathbb{C}) \simeq H_0(Y_\Gamma; \mathbb{C}) \simeq \mathbb{C},$$

where $H_c^2(Y_\Gamma; \mathbb{C})$ is compactly supported singular cohomology, the first isomorphism is a standard comparison theorem between sheaf and singular cohomology, the second follows from Poincaré duality, and the last comes from Y_Γ being connected.

More concretely, we can view $H_c^2(Y_\Gamma, \underline{\mathbb{C}})$ in terms of de Rham cohomology, so every element is represented by some 2-form on Y_Γ with compact support. Then, the trace map amounts to taking the integral over Y_Γ, i.e.,

$$\operatorname{Tr}([\omega]) = \int_{Y_\Gamma} \omega.$$

We now show that A is non-degenerate – in fact, we show that it agrees with the Petersson inner product up to constant. Using the above interpretation of the trace, we have the following concrete description of the pairing A: for $f \in \mathcal{S}_k(\Gamma)$ and $g \in \mathcal{M}_k(\Gamma)$,

$$\begin{aligned} A(f,g) &\coloneqq \int_{Y_\Gamma} \omega_f \wedge \Theta_n^{-\intercal}\overline{\omega_g} \\ &= \int_{Y_\Gamma} B\left((X - zY)^n \otimes (X - \overline{z}Y)^n\right) \cdot f(z)\overline{g(z)}\, dz \wedge d\overline{z} \\ &= \int_{Y_\Gamma} (-2iy)^n f(z)\overline{g(z)} \cdot (-2i) dx \wedge dy \\ &\overset{(n=k-2)}{=} (-2i)^{k-1} \int_{Y_\Gamma} f(z)\overline{g(z)} y^k \frac{dx\, dy}{y^2} \\ &= (-2i)^{k-1} \langle f, g \rangle_\Gamma . \end{aligned}$$

This implies that the Petersson inner product, which is non-degenerate, factors through the Eichler-Shimura map, and injectivity of the latter follows.

4. Proof of Isomorphism II: Surjectivity

To show that ES is an isomorphism, we are left to prove that the dimension over $\mathbb{R}$ on both sides are the same. Note that $\underline{L(n,\mathbb{R})}_\Gamma \otimes_{\mathbb{R}} \mathbb{C} \simeq \underline{L(n,\mathbb{C})}_\Gamma$, and so proving this for the restricted Eichler-Shimura isomorphism amounts to showing

$$2 \dim_{\mathbb{C}} \mathcal{S}_k(\Gamma) \overset{?}{=} \dim_{\mathbb{R}} H^1_!(Y_\Gamma, \underline{L(n,\mathbb{R})}_\Gamma).$$

4.1. **Dimension of Space of Cusp Forms.** We start with the left side, which we compute via Riemann-Roch. The key is to interpret cusp forms as global sections of some line bundle, which we describe by its associated divisor.

Denote $\mathcal{A}_k(\Gamma)$ as the space of automorphic forms of weight k with respect to Γ, i.e., the space of functions satisfying the definition of a modular form (of weight k w.r.t. Γ), but with the "holomorphic at the cusps" condition replaced with "meromorphic at the cusps."[2] We first want to define a divisor associated to any $0 \neq \phi \in \mathcal{A}_k(\Gamma)$. For each $p \in X_\Gamma$, we will define a valuation-at-p, denoted v_p, on $\mathcal{A}_k(\Gamma)$.

Suppose first that $p \in Y_\Gamma$. Lift p to some $p_0 \in \mathbb{H}$, and let $\lambda : \mathbb{H} \to \mathbb{D}$ (where $\mathbb{D}$ is the unit disk) be a biholomorphism sending $p_0 \mapsto 0$. If $\#\mathrm{Stab}_\Gamma(p_0) = e$, then the map locally around $p_0 \mapsto p$ given in coordinates by $z \mapsto z^e$ is a holomorphic homeomorphism, hence $t \coloneqq \lambda(z)^e$ is a uniformizer at p. We can then define

$$v_p(\phi) \coloneqq \frac{1}{e} v_{(z-p_0)}(\phi).$$

We define the valuation similarly at the cusps. If $p \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$, with lift $p_0 \in \mathbb{P}^1(\mathbb{Q})$, then we can write a Fourier series expansion $\Phi(q_h^\mu)$ for $\phi|_k\gamma$ at $i\infty$ as in §1.1, where $\gamma(i\infty) = p_0$, $q_h \coloneqq e^{2\pi i z/h}$, and $\mu = 1/2$ only when k is odd and p is an irregular cusp. Then, we naturally define

$$v_p(\phi) = \mu \cdot v_{q_h^\mu}(\Phi).$$

[2]We assume $-I_2 \notin \Gamma$ for odd k to ensure $\mathcal{A}_k(\Gamma) \neq 0$.

We can now define

$$\operatorname{div}\phi \coloneqq \sum_{p\in X_\Gamma} v_p(\phi)\cdot p, \qquad 0\neq\phi\in\mathcal{A}_k(\Gamma).$$

Let $\{q_1,\dots,q_u\}$ (resp., $\{q'_1,\dots,q'_{u'}\}$) be the regular (resp., irregular) cusps of Γ. Note that if a cusp s is irregular and k is odd, then the Fourier expansion of $f|_k\gamma$, where $\gamma\in\mathrm{SL}_2(\mathbb{Q})$ satisfying $\gamma(i\infty)=s$, is a power series in $e^{\pi inz/h}$, whereas otherwise it is a series in $e^{2\pi inz/h}$. As a result, we can realize $\mathcal{S}_k(\Gamma)$ as a subspace of $\mathcal{A}_k(\Gamma)$ via

$$\mathcal{S}_k(\Gamma)=\begin{cases}\{\phi\in\mathcal{A}_k(\Gamma):\operatorname{div}\phi\geq\sum_{j=1}^u q_j+\sum_{j=1}^{u'}q'_j\} & k\text{ even}\\ \{\phi\in\mathcal{A}_k(\Gamma):\operatorname{div}\phi\geq\sum_{j=1}^u q_j+\frac{1}{2}\sum_{j=1}^{u'}q'_j\} & k\text{ odd.}\end{cases}$$

Let K denote the field of meromorphic functions on X_Γ. As X_Γ is compact, it follows that $\mathcal{A}_k(\Gamma)$ is one-dimensional over K, meaning that given a fixed $0\neq\phi_0\in\mathcal{A}_k(\Gamma)$ and some $\phi\in\mathcal{A}_k(\Gamma)$, there exists some $f\in K$ such that $\phi=f\cdot\phi_0$. Denoting $B=\operatorname{div}\phi_0$, the above can be formulated as

$$\mathcal{S}_k(\Gamma)=\begin{cases}\{f\in K:\operatorname{div}f\geq -B+\sum_{j=1}^u q_j+\sum_{j=1}^{u'}q'_j\} & k\text{ even}\\ \{f\in K:\operatorname{div}f\geq -B+\sum_{j=1}^u q_j+\frac{1}{2}\sum_{j=1}^{u'}q'_j\} & k\text{ odd.}\end{cases}$$

Remark 4.1. One should first show that $\mathcal{A}_k(\Gamma)\neq 0$ for $k\geq 2$. We do not demonstrate this fact here, but [Shi71, Prop. 2.15] is the relevant statement.

Note that for any $f\in K$, its associated divisor $\operatorname{div}f$ cannot have non-integer coefficients, so we can take the floor $\lfloor\cdot\rfloor$ of each coefficient. If $D=\sum_p c_p\cdot p\in\mathbb{Q}\otimes_{\mathbb{Z}}\operatorname{Div}X_\Gamma$, then denote $\lfloor D\rfloor\coloneqq\sum_p\lfloor c_p\rfloor\cdot p\in\operatorname{Div}X_\Gamma$. Denoting $B'=B-\sum q_j-\sum q'_j$, we then have, in the notation of Riemann-Roch,

$$\dim_{\mathbb{C}}\mathcal{S}_k(\Gamma)=\ell(\lfloor B'\rfloor).$$

Proposition 4.2. *Suppose k is even. Let $\{\varepsilon_1,\dots,\varepsilon_r\}$ be the elliptic points of Γ, with ε_i having order e_i. Denote $\{q_j\}_{j=1}^u$ and $\{q'_j\}_{j=1}^{u'}$ as the regular and irregular cusps, respectively, as above. For any $0\neq\phi\in\mathcal{A}_k(\Gamma)$, we have*

$$\operatorname{div}\phi=\operatorname{div}\eta+\frac{k}{2}\left(\sum_{i=1}^r\left(1-\frac{1}{e_i}\right)\cdot\varepsilon_i+\sum_{j=1}^u q_j+\sum_{j=1}^{u'}q'_j\right),$$

where $\eta\coloneqq\phi(z)(dz)^{k/2}$.

Proof. It suffices to check the evaluations of v_p agree on both sides for all $p\in X_\Gamma$. We start with the non-cusps $p\in Y_\Gamma$, in which case the desired statement is

$$v_p(\phi)=v_p(\eta)+\frac{k}{2}\left(1-\frac{1}{e}\right).$$

Take $t=\lambda(z)^e$ the uniformizer at p as defined in the beginning of this subsection. Then, we have $dt/dz=e\cdot\lambda(z)^{e-1}\cdot d\lambda/dz$, and so

$$v_p(dt/dz)=\frac{1}{e}\cdot v_{z-p_0}(e\cdot\lambda(z)^{e-1}\cdot d\lambda/dz)=1-\frac{1}{e}.$$

Noting by definition of t that $v_p(dt) = 0$, we can compute

$$\begin{aligned} v_p(\eta) &= v_p(\phi(z)(dz)^{k/2}) = v_p\left(\phi(dt)^{k/2}(dz/dt)^{k/2}\right) \\ &= v_p(\phi) + \frac{k}{2}v_p(dz/dt) = v_p(\phi) - \frac{k}{2}\left(1 - \frac{1}{e}\right), \end{aligned}$$

and the desired equation follows.

Now suppose $p \in X_\Gamma - Y_\Gamma$ is a cusp. The desired statement here is $v_p(\phi) = v_p(\eta) + k/2$. Taking again γ such that $\gamma(i\infty) = p$ and $q_h = e^{2\pi i z/h}$, if we define coordinates $w = \gamma z$ and denote $\Phi(q_h)$ as the Fourier series expansion of $\phi|_k\gamma$ at $i\infty$, we get

$$\begin{aligned} \phi(w)(dw)^{k/2} &= \phi|_k\gamma(z)(dz)^{k/2} \\ &= \Phi(q)(dq)^{k/2}(dz/dq)^{k/2} \\ &= \Phi(q)(dq)^{k/2} \cdot \left(\frac{2\pi i}{h} \cdot q\right)^{-k/2}. \end{aligned}$$

Thus,

$$v_p(\eta) = v_q\left(\Phi(q)(2\pi i/h)^{-k/2}q^{-k/2}(dq)^{k/2}\right) = v_q(\Phi) - \frac{k}{2} = v_p(\phi) - \frac{k}{2},$$

and the conclusion follows. □

From this, one can deduce the following statement for all $k \geq 2$.

Corollary 4.3. *Maintain the notation as in Proposition 4.2 above. Let g be the genus of X_Γ. Then, for all $k \geq 2$ and $0 \neq \phi \in \mathcal{A}_k(\Gamma)$,*

$$\deg(\operatorname{div}\phi) = \frac{k}{2}\left((2g-2) + \sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right) + u + u'\right).$$

Proof. The case for k even is immediate from Proposition 4.2. For k odd, we can still deduce it from the above proposition via the observation $\operatorname{div}(\phi) = \frac{1}{2}\operatorname{div}(\phi^2)$. □

The following proposition has two purposes. First, it shows that $\deg(\operatorname{div}\phi) > 0$ for $0 \neq \phi \in \mathcal{A}_k(\Gamma)$. Second, the beginning construction in its proof is necessary to study the singular (in fact, simplicial) cohomology of Y_Γ and its modifications. In this spirit, we will only give the setup and general idea of the proof, leaving the details for [Shi71, Theorem 2.20].

Proposition 4.4. *Denote g as the genus of X_Γ, m the number of inequivalent cusps of Γ, and $\{e_1, \ldots, e_r\}$ the orders of the inequivalent elliptic points of Γ. Then,*

$$\frac{1}{2\pi}\int_{Y_\Gamma}\frac{dx\,dy}{y^2} = 2g - 2 + m + \sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right).$$

Note that as the left is a volume computation, it must be strictly positive, and hence Corollary 4.3 above implies that $\deg(\operatorname{div}\phi)$ is always positive.

Proof. By Radó's Theorem (see [DM68] for a proof), any compact Riemann surface can be triangulated. Indeed, we can take a basis of the homology group $H_1(X_\Gamma;\mathbb{R})$ to get a $4g$-sided polygon represented by $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$, so there are $2g$ edges given by $\{a_i,b_i\}$. Without loss of generality, this polygon can be arranged such that all elliptic points and cusps are in the interior of the polygon. One can think of this polygon as a simplicial complex on X_Γ. We can then add all elliptic points and cusps as 0-simplices, and after fixing one vertex of the polygon, we attach non-intersecting 1-simplices in both directions between the vertex and each of the elliptic points/cusps. The resulting polygon now has $4g+2m+2r$ sides. We finally add an arbitrarily small circle around each elliptic point/cusp.

Upon lifting to $\mathbb{H}^*$, we can identify this polygon with a fundamental domain $\triangle$ of Γ in $\mathbb{H}^*$. By construction, the boundary can be written in the form

$$\partial\triangle = \sum_{i=1}^{2g+m+r}(s_i - \gamma_i\cdot s_i) + \sum_{i=1}^{m+r} t_i,$$

where the s_i's correspond to the sides of the polygon, $\gamma_i\in\Gamma$ are chosen to cover all sides in the reverse direction, and the t_i's are the small circles.

Denote $\eta = y^{-1}dz$ (where $z = x_iy$), so that $d\eta = y^{-2}dx\,dy$. From the theorem statement, we are interested in computing $\int_{Y_\Gamma} d\eta$. By our above construction and Stoke's Theorem, we can write the volume as

$$\mu(Y_\Gamma) = \lim_{\text{radius}(t_i)\to 0}\int_\triangle d\eta = \lim_{\text{radius}(t_i)\to 0}\int_{\partial\triangle}\eta.$$

In terms of the boundary decomposed as above, we have that

$$\int_{\partial\triangle}\eta = \sum_{i=1}^{2g+m+r}\int_{s_i}(\eta - \eta\circ\gamma_i) + \sum_{i=1}^{m+r}\int_{t_i}\eta.$$

The main steps in the remainder of the proof rely on the following facts:

(1) For all $\gamma\in \mathrm{SL}_2(\mathbb{R})$, we have $\eta\circ\gamma - \eta = -2i\cdot d\log(j(\gamma,z))$.
(2) If $0\neq\phi\in\mathcal{A}_2(\Gamma)$ and $\xi = d(\log\phi)$, then for any $\gamma\in\Gamma$, we have $\xi\circ\gamma-\xi = 2\cdot d(\log j(\gamma,z))$.
(3) The computations relating $v_p(\phi)$ and $v_p(\phi(z)dz)$ (for $0\neq\phi\in\mathcal{A}_2(\Gamma)$) from the proof of Proposition 4.2 in the case $k=2$.

Using these, once can rewrite each summand on the right as terms appearing in the equation for Corollary 4.3. ☐

We now split into three cases: (1) $k=2$, (2) $k>2$ even, and (3) $k\geq 3$ odd. We will rely on the above computation.

$k=2$: Denote $\mathcal{M}^1 = \mathcal{M}^1_{X_\Gamma}$ as the sheaf of meromorphic differential 1-forms on X_Γ. As X_Γ is a curve, Riemann-Roch tells us that $\dim_\mathbb{C}\Gamma(X_\Gamma,\mathcal{M}^1) = g$ the genus of X_Γ. In the case $k=2$, we claim that the map $f\mapsto f\cdot dz$ is an (obviously $\mathbb{C}$-linear) isomorphism between $\mathcal{S}_2(\Gamma)\simeq\Gamma(X_\Gamma,\mathcal{M}^1)$. This is an immediate consequence of

Proposition 4.2: we have $\operatorname{div}(f(z)dz) \geq 0$ iff $\operatorname{div}(f) \geq \sum q_j + \sum q'_j$ iff $f \in \mathcal{S}_2(\Gamma)$. We conclude $\dim_{\mathbb{C}} \mathcal{S}_2(\Gamma) = g$.

$k > 2$ even: Recall we defined $B' = B - \sum q_j - \sum q'_j$, and we have $\mathcal{S}_k(\Gamma) \simeq \mathcal{L}([B'])$. We will invoke Riemann-Roch to compute the dimension of the right. We first write B' in the form of Proposition 4.2. Letting $\xi = \phi_0(dz)^{k/2}$ (where $B = \operatorname{div}\phi_0$), the proposition allows us to write

$$\begin{aligned} B' &= B - \sum_{j=1}^{u} q_j - \sum_{j=1}^{u'} q'_j = \operatorname{div}(\phi_0) - \sum_{j=1}^{u} q_j - \sum_{j=1}^{u'} q'_j \\ &= \operatorname{div}(\xi) + \frac{k}{2}\left(\sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right)\cdot \varepsilon_i + \sum_{j=1}^{u} q_j + \sum_{j=1}^{u'} q'_j\right) - \sum_{j=1}^{u} q_j - \sum_{j=1}^{u'} q'_j. \end{aligned}$$

Letting $m = u + u'$ denote the total number of inequivalent cusps and $n = k/2$ for simplicity, we can compute directly from the above

$$\begin{aligned} \deg(\lfloor B' \rfloor) &= \deg(\operatorname{div}\xi) + (n-1)m + \sum_{i=1}^{r}\left\lfloor \frac{n(e_i - 1)}{e_i} \right\rfloor \\ &= -m + n(2g - 2 + m) + \sum_{i=1}^{r}\left\lfloor \frac{n(e_i - 1)}{e_i} \right\rfloor, \end{aligned}$$

where $\deg(\operatorname{div}\xi) = n\deg(\operatorname{div}(dz)) = n(2g-2)$ as X_Γ is a curve. Noting that $\lfloor n(e_i - 1)/e_i \rfloor \geq (n-1)(e_i - 1)/e_i$, we obtain

$$\deg(\lfloor B' \rfloor) \geq 2g - 2 + (n-1)\left(2g - 2 + m + \sum_{i=1}^{r}\left(1 - \frac{1}{e_i}\right)\right).$$

By Proposition 4.4, the latter term is positive, hence $\deg\lfloor B' \rfloor > 2g - 2$. By Riemann-Roch, this forces $\ell(\lfloor B' \rfloor) = \deg\lfloor B' \rfloor - g + 1$. Substituting back $k = 2n$, we conclude

$$\dim_{\mathbb{C}} \mathcal{S}_k(\Gamma) = (k-1)(g-1) + \left(\frac{k}{2} - 1\right)m + \sum_{i=1}^{r}\left\lfloor \frac{k(e_i - 1)}{2e_i} \right\rfloor.$$

$k \geq 3$ odd: The proof is similar enough to the above case $k > 2$ even, so we do not write out all the details here. The complete proof can be found in [Shi71, p.47]. The key is to use Proposition 4.2 for $\phi = \phi_0^2$ and $\eta = \phi(dz)^k$, and note $\operatorname{div}(\phi_0) = \frac{1}{2}\operatorname{div}(\phi)$. One small but important detail is the assumption that $-I_2 \notin \Gamma$ for k odd (this, recall, was to guarantee $\mathcal{A}_k(\Gamma) \neq \{0\}$), so notably each order e_i must be odd. In the end, one can attain

$$\dim_{\mathbb{C}} \mathcal{S}_k(\Gamma) = (k-1)(g-1) + \frac{u(k-2)}{2} + \frac{u'(k-1)}{2} + \sum_{i=1}^{r}\left\lfloor \frac{k(e_i - 1)}{2e_i} \right\rfloor.$$

Note that using the exact same methods, one can compute the dimensions of $\mathcal{M}_k(\Gamma)$. It should not be of much surprise that each regular cusp contributes an extra dimension.

Explicitly, we have $\dim_{\mathbb{C}} \mathcal{M}_2(\Gamma) = g + \mathbb{1}_{m\neq 0}(m-1)$ and

$$\dim_{\mathbb{C}} \mathcal{M}_k(\Gamma) = \begin{cases} (k-1)(g-1) + \frac{k}{2}\cdot m + \sum_{i=1}^r \left\lfloor \frac{k(e_i-1)}{2e_i} \right\rfloor & k > 2 \text{ even} \\ (k-1)(g-1) + \frac{u\cdot k}{2} + \frac{u'(k-1)}{2} + \sum_{i=1}^r \left\lfloor \frac{k(e_i-1)}{2e_i} \right\rfloor & k \geq 3 \text{ odd.} \end{cases}$$

4.2. **Comparing Sheaf to Singular Cohomology.** We now turn to the other side of the Eichler-Shimura map and compute $\dim_{\mathbb{R}} H^1_*(Y_\Gamma, \underline{L(n,\mathbb{R})}_\Gamma)$ for $* = !, \emptyset$. Our first objective is to relate these sheaf cohomology groups to singular cohomology, for which we can compute the dimensions of H^1 concretely. To pass from sheaf to singular, we will use group cohomology as a middle man.

Before we traverse between sheaf cohomology and group cohomology, we take time to nod at ways to go from sheaf to singular directly, or to make the passage through different means.

The comparison between sheaf to singular in our situation can be made directly. The general phenomenon at play is the agreement between sheaf cohomology for locally constant sheaves (as is $\underline{L(n,\mathbb{R})}_\Gamma$) and singular cohomology with local coefficients, initiated in [Ste43].

Lemma 4.5. *Let L be a discrete Γ-module and $\underline{L}_\Gamma$ the locally constant sheaf on Y_Γ associated to L. Then, for all $i \geq 0$, there exists a canonical isomorphism*

$$H^i(Y_\Gamma, \underline{L}_\Gamma) \cong H^i(Y_\Gamma; L),$$

where the right is singular cohomology with local coefficients.

For a general proof of this fact, namely when the topological space is locally contractible and hereditary paracompact (meaning every open subset is paracompact), one may consult [Bel21, §4.3.11].

Proof. Because $Y_\Gamma = \Gamma\backslash\mathbb{H}$ is sufficiently nice (i.e., a metric space, hence paracompact Hausdorff), one can find a "good cover" $\mathcal{U}$ of Y_Γ such that $H^i(Y_\Gamma, \underline{L}_\Gamma) \simeq \check{H}^i(\mathcal{U}, \underline{L}_\Gamma)$. But in the Čech complex, the "good cover" means all cochains are maps from contractible open sets, hence they are valued in L. Thus, the Čech complex is exactly the cochain complex for singular cohomology with local coefficients in L. □

One could also largely bypass the discussions of group and singular cohomology by instead comparing the cohomology of locally constant sheaves with the space of **modular symbols**, which we briefly entertain now. Let $\Delta = \operatorname{Div} \mathbb{P}^1(\mathbb{Q})$ and Δ^0 its degree 0 subgroup. The space of (classical) modular symbols of weight k with respect to Γ is given by

$$\mathrm{MS}_k(\Gamma) := \operatorname{Hom}_\Gamma(\Delta^0, \operatorname{Sym}^k(\mathbb{R}^2)).$$

One of the main achievements in the work of Ash and Stevens [AS86] is summarized in the following theorem.

Theorem 4.6. *There is a natural isomorphism of $\mathbb{R}$-Hecke-modules*

$$\mathrm{MS}_k(\Gamma) \cong H^1_c(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}}(\underline{\mathbb{R}}^2)).$$

Although we do not adopt this method of approach, one could use modular symbols as a bridge between sheaf and group cohomology. Indeed, if we denote $\{p, q\}$ as the divisor $[p] - [q] \in \Delta^0$, then we have a natural map

$$\begin{aligned} \mathrm{MS}_k(\Gamma) &\to H^1(\Gamma, L(n, \mathbb{R})) \\ \xi &\mapsto \phi_\xi, \\ \phi_\xi : \gamma &\mapsto \xi(\{\infty, \gamma\infty\}). \end{aligned}$$

One may check this is indeed well-defined, and moreover, its image is contained in $H^1_!(\Gamma, L(n, \mathbb{R}))$. [Lev11, p.16]

To resume the core thread of the proof, the main bridge between sheaf cohomology and group cohomology is the following lemma, first proven in Grothendieck's Tohoku paper [Gro57] and proven concisely in [Mum74, p.23].

Lemma 4.7. *Let G be a discrete group and X be a topological space with a free discontinuous G-action such that every point has a neighborhood which is disjoint from all of its other G-orbits besides itself. Let Y be the quotient space of X by the G-action, and denote $\pi : X \twoheadrightarrow Y$ as the natural projection map. If $\mathcal{F}$ is an injective abelian sheaf on Y, then $\pi^*\mathcal{F}$ is flasque and $\Gamma(X, \pi^*\mathcal{F})$ is an injective G-module.*

In our situation, we have $G = \Gamma$, $X = \mathbb{H}$, and $\pi_\Gamma : \mathbb{H} \twoheadrightarrow Y_\Gamma$. Let L be a discrete Γ-module, and take an injective resolution $0 \to \underline{L}_\Gamma \to \mathcal{I}^\bullet$. By the lemma, $\pi_\Gamma^*\mathcal{I}^\bullet$ is a flasque resolution of $\pi_\Gamma^*\underline{L}_\Gamma$ and $\Gamma(\mathbb{H}, \pi_\Gamma^*\mathcal{I}^\bullet)$ is an injective resolution of $L = \Gamma(\mathbb{H}, \pi_\Gamma^*\underline{L}_\Gamma)$. Using the fact that $\pi_\Gamma^*\underline{L}_\Gamma = \underline{L}_\mathbb{H}$ (the constant sheaf on $\mathbb{H}$ associated to L) and $(\pi_{\Gamma,*}\underline{L}_\mathbb{H})^\Gamma = \underline{L}_\Gamma$, we can produce the following isomorphisms:

$$\begin{aligned} H^i(\Gamma, L) &= H^i\left(\Gamma(\mathbb{H}, \pi_\Gamma^*\mathcal{I}^\bullet)^\Gamma\right) \\ &\simeq H^i\left(\Gamma(Y_\Gamma, \pi_{\Gamma,*}\pi_\Gamma^*\mathcal{I}^\bullet)^\Gamma\right) \\ &\simeq H^i(\Gamma(Y_\Gamma, \mathcal{I}^\bullet)) \\ &= H^i(Y_\Gamma, \underline{L}_\Gamma). \end{aligned}$$

This applies just the same to the boundary cohomology. Given a sufficiently small open disk $U_s \subset X_\Gamma$ around a cusp $\overline{s}$ and a lift $s \in \mathbb{H}^*$ of s, there exists an open neighborhood V_s around s such that we have a holomorphic homeomorphism $\Gamma_s \backslash V_s \xrightarrow{\sim} U_s$, where Γ_s is the stabilizer of s in Γ. Thus, the lemma also gives us the isomorphism

$$H^i(U_s, \underline{L}_{\Gamma_s}) \simeq H^i(\Gamma_s, L).$$

This yields the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_!(Y_\Gamma, \underline{L}_\Gamma) & \longrightarrow & H^1(Y_\Gamma, \underline{L}_\Gamma) & \longrightarrow & H^1_\partial(Y_\Gamma, \underline{L}_\Gamma) \\ & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ 0 & \longrightarrow & \ker\left(\bigoplus_{\overline{s}\in\Gamma\backslash\mathbb{P}^1(\mathbb{Q})} \mathrm{res}_{\Gamma,\Gamma_s}\right) & \longrightarrow & H^1(\Gamma, L) & \xrightarrow{\oplus\,\mathrm{res}_{\Gamma,\Gamma_s}} & \bigoplus_{\overline{s}\in\Gamma\backslash\mathbb{P}^1(\mathbb{Q})} H^1(\Gamma_s, L) \end{array}$$

It follows that the dashed vertical arrow is also an isomorphism.

For our computational purposes, it is beneficial to describe the kernel in more explicit terms. First, as the stabilizer of any cusp s in $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to $\mathbb{R} \times \{\pm I_2\}$, we deduce that Γ_s is isomorphic, at least in $\mathrm{PSL}_2(\mathbb{Q})$, to $\mathbb{Z}$. Working over $\mathrm{PSL}_2(\mathbb{Q})$ if necessary, we can write $\Gamma_s = \langle \pi_s \rangle$ for some $\pi_s \in \Gamma_s$. It follows that every parabolic element of Γ is conjugate to a power of some π_s, so we may reduce the subset of parabolic elements to a set of "representatives," which we give by $P \coloneqq \{\pi_s : s \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})\}$.

We now consider the boundary part of the group cohomology exact sequence above. Note from the definition of a 1-cocycle that any $\phi \in H^1(\Gamma_s, L)$ will satisfy, for all $m \in \mathbb{Z}$,

$$\phi(\pi_s^m) = (1 + \pi_s + \cdots + \pi_s^{m-1})\phi(\pi_s),$$

hence ϕ is determined solely by its value on π_s. This yields $Z^1(\Gamma_s, L) = L$. For the coboundaries, we note that if $\phi(\pi_s) = (\pi_s - 1)a$ for some $a \in L$, then

$$\phi(\pi_s^m) = (1 + \cdots + \pi_s^{m-1})\phi(\pi_s) = (1 + \cdots + \pi_s^{m-1})(\pi_s - 1)a = (\pi_s^m - 1)a,$$

so $B^1(\Gamma_s, L) = (\pi_s - 1)L$.

It is clear from here that the kernel, which we call **parabolic cohomology** and denote by $H^1_P(\Gamma, L)$, is the subspace of $H^1(\Gamma, L)$ consisting of all (equivalence classes of) 1-cocycles ϕ such that $\phi(\pi_s) \in (\pi_s - 1)L$ for all cusps s.

Remark 4.8. Note from the above computation of $\phi(\pi_s^m)$, which we can extend to all conjugates $\phi(\gamma\pi_s^m\gamma^{-1})$, that we could have considered all parabolic elements and not just the select subset P. In other words, we also have

$$H^1_P(\Gamma, L) = \{[\phi] \in H^1(\Gamma, L) : \phi(\pi) \in (\pi - 1)L \text{ for all } \pi \text{ parabolic}\}.$$

We now pass from group cohomology to singular (rather, simplicial) cohomology. Denote $Y_\Gamma^{(0)}$ as the open Riemann surface obtained by removing from X_Γ small, disjoint open disks around each cusp, and denote $\mathbb{H}^{(0)}$ as the preimage of $Y_\Gamma^{(0)}$ under the natural projection $\mathbb{H}^* \twoheadrightarrow \Gamma\backslash\mathbb{H}^*$. Then, we can follow the beginning of the proof of Proposition 4.4 to construct a simplicial complex K of $\mathbb{H}^{(0)}$ satisfying the following conditions:

- (S1) K is stable under the Γ-action.
- (S2) For every cusp $s \in S$, there exists a 1-chain t_s of K which maps onto the boundary of the excluded disk around s.
- (S3) There exists a fundamental domain $\triangle^{(0)}$ for the Γ-action in $\mathbb{H}^{(0)}$ whose closure consists of finitely many simplices in K.

With such a simplicial complex K, we can construct the exact sequence in singular cohomology analogous to the one in group cohomology defining the parabolic cohomology group $H^1_P(\Gamma, L)$.

For $i \in \{0, 1, 2\}$, let A_i be the $\mathbb{R}$-vector space generated by the i-simplices in K. Attached to K then is the natural chain complex

$$0 \to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \xrightarrow{\mathfrak{a}} \mathbb{R} \to 0,$$

where ∂ is the usual boundary operator and $\mathfrak{a}$ is the augmentation map given by adding the coefficients of the 0-simplices. But this is a free $\mathbb{R}[\Gamma]$-module resolution of $\mathbb{R}$, so after taking the corresponding cochain complex with coefficients in L, we have a canonical isomorphism $H^1(K;L) \cong H^1(\Gamma, L)$.

We treat the "boundary part" of the cohomology in a similar spirit. Denote now P as the set of all parabolic elements in Γ. (This should not raise any concerns thanks to Remark 4.8.) Take any $\pi \in P$; we established before that $\pi = \gamma\pi_s\gamma^{-1}$ for some cusp s and $\gamma \in \Gamma$. Property (K2) dictates that $\gamma(t_s)$ must be a 1-simplex of K. The image of $\gamma(t_s)$ in $\triangle^{(0)}$ has universal covering space isomorphic to $\mathbb{R}$, which we embed by ι_π into the universal cover $\mathbb{H}$ of $\triangle^{(0)}$. By definition, the closure $\overline{\gamma(t_s)}$ gives a triangulation of a fundamental domain of $\Gamma_\pi \backslash \iota_\pi(\mathbb{R})$. We can now define $A_i(\pi)$, for $i \in \{0, 1\}$, to be the free $\mathbb{R}[\Gamma_\pi]$-module generated by the i-simplices in $\overline{\gamma(t_s)}$, giving us the free resolution of $\mathbb{R}[\Gamma_\pi]$-modules

$$0 \to A_1(\pi) \xrightarrow{\partial} A_0(\pi) \xrightarrow{\mathfrak{a}} \mathbb{R} \to 0.$$

The inclusion ι_π induces a natural inclusion of chain complexes $A_\bullet(\pi) \hookrightarrow A_\bullet$, which collectively gives $\bigoplus_{\pi\in P} A_\bullet(\pi) \to A_\bullet$. It is a standard fact that free (even better, projective) resolutions are unique up to homotopy equivalence. Consequently, if $F_\bullet$ (resp., $F_\bullet(\pi)$) is the standard $\mathbb{R}[\Gamma]$-free (resp., $\mathbb{R}[\Gamma_\pi]$-free) resolution of $\mathbb{R}$, then we get induced homotopy equivalences between the various complexes such that the following diagram commutes:

$$\begin{array}{ccc} A_\bullet & \longleftarrow & \bigoplus_{\pi\in P} A_\bullet(\pi) \\ \downarrow & & \downarrow \\ F_\bullet & \longleftarrow & \bigoplus_{\pi\in P} F_\bullet(\pi) \end{array}$$

Applying $\operatorname{Hom}_{\mathbb{R}[\Gamma]}(-, L)$ everywhere and taking cohomology gives us the desired commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1_P(\Gamma, L) & \longrightarrow & H^1(\Gamma, L) & \longrightarrow & \bigoplus_{\pi\in P} H^1(\Gamma_\pi, L) \\ & & & & \downarrow \simeq & & \downarrow \simeq \\ & & & & H^1(K; L) & \longrightarrow & \bigoplus_{\pi\in P} H^1(K(\pi); L) \end{array}$$

where $H^\bullet(K(\pi), L)$ is understood from the construction to be the cohomology groups of the cochain complex $\operatorname{Hom}_{\mathbb{R}[\Gamma_\pi]}(A_\bullet(\pi), L)$. We denote the kernel of the bottom map as $H^1_P(K; L)$, and it is evident that there is a canonical isomorphism $H^1_P(K; L) \cong H^1_P(\Gamma, L)$.

It now remains to compute $\dim_{\mathbb{R}} H^1_P(K; L)$ and check it agrees with $2\dim_{\mathbb{C}} \mathcal{S}_k(\Gamma)$ when $L = L(n, \mathbb{R})$. Our proceeding computations will heavily rely on the explicit construction of the simplicial complex K as explained in the proof of Proposition 4.4.

Remark 4.9 (Confession). We did not have time to finish these arguments. However, we do provide a complete proof of surjectivity in §6, which in the end is roughly the same argument. There, the proof is slicker because we only concern ourselves with regular cusps. For the remaining arguments using the above approach, look at the two propositions and the corollary in [Hid93, §6.1].

5. Modular Forms as Global Sections

We will now give a purely cohomological construction of the Eichler-Shimura map. Three advantages of this reformulation are (1) it is less ad-hoc, (2) the dimension-counting arguments as done in the previous section become much easier using results from cohomology, and (3) one can naturally replace Zariski cohomology with étale cohomology and begin to construct Galois representations, as is done in [Del71].

The entry point is that, for instance, a cusp form $f(z)$ of weight 2 and level N can be seen as a holomorphic 1-form $f(z)\,dz$ on the modular curve $X_1(N)$ which vanishes at the cusps. In other words, we can realize $f \in H^0(X_1(N), \Omega^1_{X_1(N)}(\text{cusps}))$. We can replace Ω^1 with any k^{th}-tensor power to produce cusp forms of weight $2k$.

The obvious pitfall here is that we cannot recover modular forms of odd weight in this manner. Somehow, we need to make sense of "1/2"-forms.

The beautiful answer comes from deformation theory. By seeing the dual of $\Omega^1_{X_1(N)}$ as the (sheaf of sections of the) tangent bundle of $X_1(N)$, and noting that $X_1(N)$ is a fine moduli space of elliptic curves with some level structure [KM85], one can study the deformation of elliptic curves and produce a version of the Kodaira-Spencer map, which will be of the form $\mathrm{KS} : \omega^{\otimes 2} \simeq \Omega^1_{X_1(N)}$ for some line bundle ω on $X_1(N)$. We will then be able to define modular forms of weight k for $\Gamma_1(N)$ as global sections of $\omega^{\otimes k}$.

5.1. Relative Elliptic Curves.

By the principle described above, we will be working in the relative setting, primarily with elliptic curves over an arbitrary analytic space. We define this explicitly.

Definition 5.1 (S-elliptic curve)**.** Let S be a complex analytic space. An **elliptic curve over S** is a proper flat morphism $f : E \to S$ of analytic spaces such that its fibers are complex elliptic curves and it comes equipped with a section $e : S \to E$, a group law over S, and an inverse S-morphism which make it an S-group object.

Before we proceed, we recall some important results of higher direct image sheaves. The following analytic version of the cohomology and base change theorem (see [Har77, Theorem 12.11] for the algebraic version) will be useful for us.

Theorem 5.2 (Cohomology and Base Change)**.** *Let $f : X \to Y$ be a proper morphism of analytic spaces, and let $\mathcal{F} \in \mathrm{Coh}(X)$ be flat over Y. Denote $X_y := f^{-1}(y)$ for the fiber above y and $\mathcal{F}_y := \mathcal{F}|_{X_y}$. For $y \in Y$, consider the natural map*

$$\varphi^i(y) : R^i f_*(\mathcal{F})_y \otimes_{\mathcal{O}_{Y,y}} k(y) \to H^i(X_y, \mathcal{F}_y).$$

(1) If $\varphi^i(y)$ is surjective, then it is an isomorphism.

(2) If $\varphi^i(y)$ is surjective for all $y \in Y$, then $\varphi^{i-1}(y)$ is surjective if and only if $R^i f_* \mathcal{F}$ is locally free on some open neighborhood $U \ni y$. In this case, $R^i f_* \mathcal{F}$ is of formation compatible with arbitrary analytic base change over U.

We most frequently use this theorem to check that two derived pushforwards are isomorphic by reducing the statement to a standard isomorphism on cohomology (e.g., Poincaré duality) via taking stalks. This is exactly what we do for the following corollary.

Corollary 5.3. *Let $f : X \to Y$ be a proper flat morphism of analytic spaces with connected reduced fibers. Then, the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

Proof. As f is proper, its fibers X_y are compact, which means we have

$$(R^0 f_*\mathcal{O}_X)_y \simeq H^0(X_y, \mathcal{O}_X|_{X_y}) = \mathbb{C} \simeq \mathcal{O}_{Y,y},$$

the first isomorphism following from the theorem above. □

Let $f : E \to S$ be an elliptic curve over S. The relative Hodge complex gives us the short exact sequence

$$0 \to f^{-1}\mathcal{O}_S \to \mathcal{O}_E \xrightarrow{d} \Omega^1_{E/S} \to 0.$$

The sequence is exact intuitively because $h \in \ker d$ if and only if h is constant on each fiber, which is exactly the sections of the inverse image sheaf $f^{-1}\mathcal{O}_S$. Note $d : \mathcal{O}_E \to \Omega^1_{E/S}$ is surjective only because the fibers of f are elliptic curves, hence $\Omega^2_{E/S} = 0$. By taking the long exact sequence for derived pushforwards, we get the exact sequence

$$0 \to f_*f^{-1}\mathcal{O}_S \to f_*\mathcal{O}_E \to f_*\Omega^1_{E/S} \to R^1 f_*(f^{-1}\mathcal{O}_S) \to R^1 f_*\mathcal{O}_E.$$

Because f is surjective, we have $f_*f^{-1}\mathcal{O}_S \simeq \mathcal{O}_S \simeq f_*\mathcal{O}_E$, where the latter isomorphism is from Corollary 5.3. Thus, we can truncate the exact sequence and are left with

$$0 \to f_*\Omega^1_{E/S} \to R^1 f_*(f^{-1}\mathcal{O}_S) \to R^1 f_*\mathcal{O}_E.$$

Since $f_*f^{-1}\mathcal{O}_S \simeq \mathcal{O}_S$, the relative Poincaré cup product gives us a natural map $R^1 f_*\underline{\mathbb{R}} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_S \simeq R^1 f_*\underline{\mathbb{C}} \otimes_{\underline{\mathbb{C}}} f_*(f^{-1}\mathcal{O}_S) \xrightarrow{\cup} R^1 f_*(f^{-1}\mathcal{O}_S)$. But on each fiber $E_s := f^{-1}(s)$, this cup product is just

$$H^1(E_s, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{S,s} \to H^1(E_s, (f^{-1}\mathcal{O}_S)|_{E_s}) = H^1(E_s, \underline{\mathcal{O}_{S,s}})$$

which is an isomorphism. Hence, we can replace $R^1 f_*(f^{-1}\mathcal{O}_S)$ with $R^1 f_*\underline{\mathbb{R}} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_S$. This leaves us with the exact sequence

$$0 \to f_*\Omega^1_{E/S} \to R^1 f_*\underline{\mathbb{R}} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_S \to R^1 f_*\mathcal{O}_E.$$

This exact sequence on its fibers is just

$$0 \to H^0(E_s, \Omega^1_{E_s}) \to H^1(E_s, \mathbb{C}) \to H^1(E_s, \mathcal{O}_{E_s}),$$

which by the Hodge decomposition is a split exact sequence. This means our relative exact sequence is actually a short exact sequence. We can also replace the last term $R^1 f_*\mathcal{O}_E$ with $f_*(\Omega^1_{E/S})^\vee$ by Serre duality on the fibers.

Denote $\omega = \omega_S := f_*\Omega^1_{E/S}$. We can write our short exact sequence in terms of ω as follows:

$$0 \to \omega \to R^1 f_*\underline{\mathbb{R}} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_S \xrightarrow{q} \omega^{-1} \to 0. \tag{1}$$

Remark 5.4. Denoting $e : S \to E$ as the identity section, we have a natural map $\Omega^1_{E/S} \to (e \circ f)_* \Omega^1_{E/S} = e_* f_* \Omega^1_{E/S}$ induced by the S-morphism $e \circ f : E \to E$. The adjunction between inverse and direct image gives a natural map $e^* \Omega^1_{E/S} \to f_* \Omega^1_{E/S}$. But all stalks on both sides are one-dimensional over $\mathbb{C}$, hence this is an isomorphism. We may interpret $e^* \Omega^1_{E/S}$ as the cotangent space of E at the identity, so ω^{-1} is understood to be the relative Lie algebra $\underline{\mathrm{Lie}}_S(E)$.

We will investigate the middle term of this short exact sequence, which we can rewrite as $R^1 f_* \underline{\mathbb{Z}} \otimes_{\underline{\mathbb{Z}}} \mathcal{O}_S$. The first property to prove is that $R^1 f_* \underline{\mathbb{Z}}$ is a local system (i.e., a locally constant sheaf, defined below) of rank 2 free $\mathbb{Z}$-modules.

Definition 5.5 (Local System)**.** Let R be a ring and $\underline{R}$ the associated constant sheaf. A **local system** of R-modules is a $\underline{R}$-module M which is locally constant. (If $R = \mathbb{Z}[\Gamma]$ is a group ring, we say Γ-module instead of $\mathbb{Z}[\Gamma]$-module for simplicity.)

Proposition 5.6. *Let $f : X \to S$ be a proper smooth map with connected fibers of dimension d. For any torsion-free abelian group G, the sheaf $R^i f_* \underline{G}$ is a local system of G-modules and compatible with base change for $i \leq 2d$, and it vanishes for $i > 2d$. Furthermore, the natural map $\underline{G} \otimes R^{2d} f_* \underline{\mathbb{Z}} \to R^{2d} f_* \underline{G}$ is an isomorphism and $R^{2d} f_* \underline{\mathbb{Z}}$ is a local system of rank 1 free $\mathbb{Z}$-modules.*

Remark 5.7. It is clear (at least for $G = \mathbb{Z}$) that $\underline{G} \simeq f_* \underline{G}$, and then assuming the locally constant statement, the last statements clearly follow from Poincaré duality. The other statements crucially make use of Theorem 5.2.

In fact, we can make our life easier by restricting S to be smooth, in which case the result is clear from Ehresmann's fibration theorem that f is locally trivial. One can prove this by reducing to the case where S is smooth by working locally, see [Con05, Theorem 1.2.1.6].

By construction, $R^1 f_* \underline{\mathbb{Z}}$ is a $\underline{\mathbb{Z}}$-module, and the proposition confirms it is locally constant. In special cases, such as Example 5.11 below, it is actually a constant sheaf. One such class of examples, which gives context to §5.3 below, comes from the following useful perspective of local systems.

Theorem 5.8. *Let X be a topological space with a universal cover $\pi : \widetilde{X} \to X$.*[3] *Fix some $x \in X$, and denote $G = \pi_1(X, x)$. Let R be a ring. There is an equivalence of categories between the R-modules with an R-linear G-action and local systems of R-modules, where an G-R-module M corresponds to the sheaf of continuous sections of the natural projection $(\widetilde{X} \times M)/G \twoheadrightarrow X$.*

Remark 5.9. By this characterization, it is clear that the sheaf $\underline{L(n, \mathbb{R})}_\Gamma$ on Y_Γ is a local system of free $\mathbb{R}$-modules of rank $n + 1$.

[3]Assuming X is connected, locally path-connected, and semi-locally simply connected is sufficient, for instance.

5.2. **Kodaira-Spencer Map.** The final part of our general discussion of S-elliptic curves is to relate ω, which by definition is $f_*\Omega^1_{E/S}$, to Ω^1_S.

Consider an S-elliptic curve $f : E \to S$. We have the following fundamental exact sequence of differentials

$$0 \to f^*\Omega^1_S \to \Omega^1_E \to \Omega^1_{E/S} \to 0,$$

where implicitly the latter two are differentials over the base field $\mathbb{C}$. Dualizing gives us

$$0 \to (\Omega^1_{E/S})^\vee \to (\Omega^1_E)^\vee \to (f^*\Omega^1_S)^\vee \to 0$$

The long exact sequence for the derived pushforward gives us connecting map $\delta : f_*(f^*\Omega^1_S)^\vee \to R^1 f_*(\Omega^1_{E/S})^\vee$. We can thus consider the following composition

$$f_*(\Omega^1_{E/S}) \otimes f_*(f^*\Omega^1_S)^\vee \xrightarrow{\mathrm{id}\otimes\delta} f_*(\Omega^1_{E/S}) \otimes R^1 f_*(\Omega^1_{E/S})^\vee \xrightarrow{\cup} R^1 f_*(\mathcal{O}_E) \simeq f_*(\Omega^1_{E/S})^\vee,$$

where the last isomorphism is from relative version of Serre duality. (Once again, the isomorphism follows from the usual Serre duality on the fibers.) Recalling $\omega := f_*(\Omega^1_{E/S})$, this leaves us with

$$\omega \otimes f_* f^*(\Omega^1_S)^\vee \to \omega^{-1}.$$

But note that the middle term is just $\Omega^{1\vee}_S$ since f is proper and smooth with connected fibers. (In particular, (1) $\mathcal{O}_S \simeq f_*\mathcal{O}_E$ and (2) Ω^1_S is locally free of finite rank.) Thus, we are left with $\omega \otimes \Omega^\vee_S \to \omega^{-1}$, which yields the **Kodaira-Spencer map**

$$\underline{\mathrm{KS}}_{E/S} : \omega_S^{\otimes 2} \to \Omega^1_S. \tag{2}$$

5.3. **Universal Elliptic Curve.** In light of the Kodaira-Spencer map, our immediate objectives are self-explanatory:

(1) Construct an elliptic curve over $S = X_\Gamma := \Gamma\backslash\mathbb{H}^*$.
(2) Show in this case that $\underline{\mathrm{KS}}$ is an isomorphism (at least, up to twisting by the divisor of cusps).

The first point will really be divided into three steps:

(a) Construct an elliptic curve over $S = \mathbb{H}$.
(b) Descend the short exact sequence (1) and the Kodaira-Spencer map (2) from $S = \mathbb{H}$ to $S = Y_\Gamma := \Gamma\backslash\mathbb{H}$.
(c) Extend both to the compactification X_Γ.

We naturally begin with (1a). We employ the fact that $\mathbb{H}$ (in particular, $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$) parametrizes complex elliptic curves $\mathbb{C}/\langle 1, \tau\rangle$ by the variable $\tau \in \mathbb{H}$ to construct a relative elliptic curve over $\mathbb{H}$, which we denote $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathbb{H}$.

Definition 5.10 (Elliptic Curve over $\mathbb{H}$)**.** We construct $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathbb{H}$ in three steps.

(1) Consider the lattice

$$\Lambda = \mathbb{Z}^2 \times \mathbb{H} \hookrightarrow \mathbb{C} \times \mathbb{H}$$
$$((m, n), z) \mapsto (m + nz, z).$$

(2) Consider the quotient top space $(\mathbb{C} \times \mathbb{H})/\Lambda$ where $(\tau, z) \simeq (\tau + \lambda, z)$ for $\lambda \in \Lambda_z$.

(3) The projection map $\pi : \mathbb{C} \times \mathbb{H} \to (\mathbb{C} \times \mathbb{H})/\Lambda$ is a priori just a local $\mathcal{C}^\infty$-isomorphism, but one can endow $\mathbb{C} \times \mathbb{H}$ with a unique complex manifold structure such that π is a local biholomorphism. We denote $\mathbb{C} \times \mathbb{H}$ with the complex structure as $\mathcal{E}^{\mathrm{an}}$.

As mentioned before, when S is connected (as is the case for $S = \mathbb{H}$), the higher direct image $R^1 f_* \underline{\mathbb{Z}}$ must be constant. As we will soon care about an explicit isomorphism $R^1 f_* \underline{\mathbb{Z}} \xrightarrow{\simeq} \underline{\mathbb{Z}}^2$, we construct such an isomorphism α^{an} in the case $f = f^{\mathrm{an}}$. To define such isomorphisms, we will fix once and for all $i = \sqrt{-1} \in \mathbb{C}$.

Example 5.11 ($R^1 f_*^{\mathrm{an}} \underline{\mathbb{Z}}$ is constant)**.** We will construct a map $\alpha^{\mathrm{an}} : R^1 f_*^{\mathrm{an}} \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}^2$. The key is to identify $\mathbb{Z}^2 \simeq H^1(\mathbb{C}/\langle 1, i\rangle, \mathbb{Z})$; note that this requires a choice of basis for $\mathbb{Z}^2$, as canonically from Poincaré duality we have $\mathbb{Z}^2 \simeq H_1(\mathbb{C}/\langle 1, i\rangle, \mathbb{Z}) \simeq H^1(\mathbb{C}/\langle 1, i\rangle, \mathbb{Z})^\vee$. We have a $\mathcal{C}^\infty$-isomorphism

$$(x + iy, z) \longmapsto (x + zy, z)$$

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{H} & \xrightarrow[\mathcal{C}^\infty \text{ iso.}]{} & \mathbb{C} \times \mathbb{H} \\ & \searrow \quad \swarrow & \\ & \mathbb{H} & \end{array}$$

which induces a $\mathcal{C}^\infty$-isomorphism $(\mathbb{C}/\langle 1, i\rangle) \times \mathbb{H} \to \mathcal{E}^{\mathrm{an}}$ over $\mathbb{H}$, as shown below.

$$\begin{array}{ccc} (\mathbb{C}/\langle 1, i\rangle) \times \mathbb{H} & \xrightarrow{\mathcal{C}^\infty \text{ iso}} & (\mathbb{C} \times \mathbb{H})/\Lambda \\ {\scriptstyle \mathrm{pr}_{\mathbb{H}}} \downarrow & & \downarrow {\scriptstyle f^{\mathrm{an}}} \\ \mathbb{H} & \xrightarrow{\mathrm{id}} & \mathbb{H} \end{array}$$

The push-pull map [Vak25, §2.7.4] gives us a natural map $f_*^{\mathrm{an}} \underline{\mathbb{Z}} \to \mathrm{pr}_{\mathbb{H},*} \underline{\mathbb{Z}}$, which further induces a map on the derived pushforward $R^1 f_*^{\mathrm{an}} \underline{\mathbb{Z}} \to R^1 \mathrm{pr}_{\mathbb{H},*} \underline{\mathbb{Z}}$. But on fibers, the latter is canonically isomorphic to $H^1(\mathbb{C}/\langle 1, i\rangle, \mathbb{Z}) \simeq (\mathbb{Z}^2)^\vee$ everywhere, so we have the map $R^1 f_*^{\mathrm{an}} \underline{\mathbb{Z}} \to (\underline{\mathbb{Z}}^2)^\vee$, which we can easily check on stalks to be an isomorphism.

The last minor step is to specify an isomorphism $(\mathbb{Z}^2)^\vee \simeq \mathbb{Z}^2$. Following [Del71, §2.1], we will select an orientation-*reversing* isomorphism so that the composition $\alpha^{\mathrm{an}} : R^1 f_*^{\mathrm{an}} \underline{\mathbb{Z}} \xrightarrow{\simeq} (\underline{\mathbb{Z}}^2)^\vee \simeq \underline{\mathbb{Z}}^2$ makes the following diagram *anti*-commutative:

$$\begin{array}{ccc} \bigwedge^2 R^1 f_*^{\mathrm{an}} \underline{\mathbb{Z}} & \xrightarrow{\bigwedge^2 \alpha^{\mathrm{an}}} & \bigwedge^2 \underline{\mathbb{Z}}^2 \\ {\scriptstyle \cup} \downarrow {\scriptstyle \simeq} & & \downarrow {\scriptstyle \simeq} \\ R^2 f_*^{\mathrm{an}} \underline{\mathbb{Z}} & \xrightarrow{\simeq} & \underline{\mathbb{Z}} \end{array}$$

Remark 5.12. The right vertical arrow is, given a choice of basis $\{e_1, e_2\}$ of $\mathbb{Z}^2$, the map $e_1 \wedge e_2 \mapsto 1$, the left vertical arrow is the relative cup product which one can check is an isomorphism on its stalks, and the bottom map is the canonical map from Proposition 5.6.

Definition 5.13 (Permitted Isomorphism)**.** Following Deligne, we say an isomorphism $\alpha : R^1 f_* \mathbb{Z} \to \underline{\mathbb{Z}}^2$ for some relative elliptic curve f is **permitted** if it satisfies the above anti-commutative diagram.

We maintain a fixed $i = \sqrt{-1} \in \mathbb{C}$, and we define $\mathbb{H}$ such that $i \in \mathbb{H}$. Consider the functor $\mathsf{AnSp}_{\mathbb{C}} \to \mathsf{Set}$ which associates to a complex analytic space S the set of pairs $(f : E \to S, \alpha : R^1 f_* \underline{\mathbb{Z}} \xrightarrow{\simeq} \underline{\mathbb{Z}}^2)$ where f gives an S-elliptic curve and α is permissible. Our construction $(f^{\mathrm{an}}, \alpha^{\mathrm{an}})$ is significant in the following way.

Proposition 5.14. *This functor is representable by* $(f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathbb{H}, \alpha^{\mathrm{an}} : R^1 f^{\mathrm{an}}_* \underline{\mathbb{Z}} \xrightarrow{\simeq} \underline{\mathbb{Z}}^2)$*.*

Remark 5.15. One can define a similar functor for relative complex tori of a specified relative dimension, where the associated set must also carry the data of a principal polarization on the complex torus. Such a functor will always be representable, see [Con05, Theorem 1.4.3.1].

Remark 5.16. Sometimes we will refer to this universal object abstractly as $f : E_X \to X$ for more natural notation, but for the proceeding, it is always kosher to choose a base point i and work with the coordinate-specified version f^{an}.

We wish to study the short exact sequence (1) and the Kodaira-Spencer map (2) for this universal relative elliptic curve $(f : E_X \to X, \alpha_X) = (f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathbb{H}, \alpha^{\mathrm{an}})$. The map α^{an} gives us a de facto isomorphism $R^1 f^{\mathrm{an}}_* \underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}}^2$. In this setting, the Kodaira-Spencer map will also be an isomorphism.

In fact, it is equivariant under $\mathrm{SL}_2(\mathbb{R})$, whose action we now describe. The $\mathrm{SL}_2(\mathbb{R})$-action on Ω^1_X comes from the natural action on $X = \mathbb{H}$. For ω, the universal property of $(f : E_X \to X, \alpha_X)$ means that the isomorphism $\gamma \circ \alpha_X : R^1 f_* \underline{\mathbb{R}} \xrightarrow{\simeq} \underline{\mathbb{R}}^2 \xrightarrow{\gamma} \underline{\mathbb{R}}^2$ for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$ induces an $\mathrm{SL}_2(\mathbb{R})$-action on $R^1 f_* \underline{\mathbb{R}}$, which we can then descend to ω^{-1} via the surjection $R^1 f_* \underline{\mathbb{R}} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_X \to \omega^{-1}$ from the short exact sequence. Taking the standpoint that modular forms of weight k should be global sections of $\omega^{\otimes k}$, we have that this $\mathrm{SL}_2(\mathbb{R})$-action gives exactly the automorphy condition for modular forms, see [Del71, §2.3].

Theorem 5.17. *For the universal elliptic curve $E_X \to X$, the Kodaira-Spencer map $\underline{\mathrm{KS}}_{E_X/X} : \omega_X^{\otimes 2} \to \Omega^1_X$ is an $\mathrm{SL}_2(\mathbb{R})$-equivariant isomorphism.*

Almost tautologically, the short exact sequence (1) becomes $\mathrm{SL}_2(\mathbb{R})$-equivariant as well. For a complete description of the $\mathrm{SL}_2(\mathbb{R})$-action on each sheaf and the equivariance of the short exact sequence, consult [Con05, Lemma 1.5.4.4].

5.4. **Descent to Modular Curves.** From this $\mathrm{SL}_2(\mathbb{R})$-equivariance, one expects to produce both the short exact sequence and the Kodaira-Spencer map (as an isomorphism) for the relative elliptic curve over the base space $Y_\Gamma := \Gamma \backslash \mathbb{H}$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group of the first kind (meaning $\Gamma \backslash \mathbb{H}^*$ is compact, see [Shi71, §1.5]). Indeed, one can perform such a descent along $\mathbb{H} \to Y_\Gamma$ via Grothendieck's theory of descent, which we describe in the topological setting.

Definition 5.18 (Γ-sheaf)**.** Let $\pi : X \to S$ be a covering map of topological spaces, and let Γ be a group of (right) S-automorphisms of X which acts transitively on fibers and freely on X. A **Γ-sheaf** on X is a sheaf $\mathcal{F}$ equipped with isomorphisms $\alpha_\gamma : \gamma^*\mathcal{F} \xrightarrow{\simeq} \mathcal{F}$ for each $\gamma \in \Gamma \subset \mathrm{Aut}_S(X)$ which satisfy the relations $\alpha_\gamma \circ \gamma^*\alpha_\sigma = \alpha_{\gamma\sigma}$. Note this endows a (left) Γ-action on $\mathcal{F}$.

Theorem 5.19. *Take the setup* $(\pi : X \to S, \Gamma \subset \mathrm{Aut}_S(X))$ *as in the above definition. There is an equivalence of categories between* Γ*-sheaves of sets on* X *and sheaves of sets on* S*, where* $\mathcal{G} \mapsto \pi^*\mathcal{G}$ *and* $\mathcal{F} \mapsto \pi_*(\mathcal{F})^\Gamma$ *(the* Γ*-invariant sections of* $\pi_*(\mathcal{F})$*).*

Denote $f_\Gamma : \mathbb{H} \to \Gamma\backslash\mathbb{H} = Y_\Gamma$, and let $\omega_{Y_\Gamma} := f_{\Gamma,*}(\omega_X)^\Gamma$. Applying this to $\underline{\mathbb{R}}^2$ gives us a local system $\mathcal{U}_{Y_\Gamma}$ of 2-dimensional $\mathbb{R}$-vector spaces on Y_Γ. We thus have a sequence

$$0 \to \omega_{Y_\Gamma} \to \mathcal{U}_{Y_\Gamma} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_{Y_\Gamma} \to \omega_{Y_\Gamma}^{-1} \to 0. \tag{3}$$

This is clearly left-exact because both the direct image and taking Γ-invariants are in general left-exact. Right-exactness follows from the $\mathrm{SL}_2(\mathbb{R})$-equivariance, so taking the Γ-invariants should preserve surjectivity on the stalks. In a similar vein, we descend the Kodaira-Spencer map to an isomorphism $\underline{\mathrm{KS}}_{Y_\Gamma} : \omega_{Y_\Gamma}^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{Y_\Gamma}$.

To complete our reformulation of modular forms, we need to extend the Kodaira-Spencer map to the cusps, i.e., define it for the base space $X_\Gamma := \Gamma\backslash\mathbb{H}^*$. It turns out that one can only do this when all cusps of X_Γ are regular, as defined below.

Definition 5.20 (Regular, Irregular)**.** Let $s \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ be a cusp, and $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma s = i\infty$. Then, we have by definition

$$\{\pm 1\} \cdot (\gamma\Gamma_x\gamma^{-1}) = \{\pm 1\} \cdot \begin{pmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{pmatrix}$$

and hence $\gamma\Gamma_x\gamma^{-1} = \begin{pmatrix} \varepsilon & h \\ 0 & \varepsilon \end{pmatrix}^{\mathbb{Z}}$ for $\varepsilon \in \{\pm 1\}$. We say s is **regular** if $\varepsilon = +1$ and **irregular** if $\varepsilon = -1$.

If there are no irregular cusps in X_Γ, then we can extend the Kodaira-Spencer isomorphism by twisting the sheaf of differentials by the divisor of cusps.

Theorem 5.21. *Let* $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ *be a Fuchsian group of the first kind such that* X_Γ *has no irregular cusps. Then, the Kodaira-Spencer map* $\underline{\mathrm{KS}}_{Y_\Gamma}$ *induces an isomorphism*

$$\underline{\mathrm{KS}}_{X_\Gamma} : \omega_{X_\Gamma}^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{X_\Gamma}(\mathrm{cusps}).$$

For a complete proof, including a discussion of regular cusps, see [Con05, §1.5.6-7]. Intuitively, extending to the cusps involves working locally at the cusp. Taking the cusp at infinity, for instance, our local coordinates are $q = e^{2\pi i z}$, so $dq/q = 2\pi i\, dz$. This indicates that we should allow for simple poles at the cusps.

5.5. **Modular Forms and the Eichler-Shimura Map.** We are now ready to define modular forms cohomologically and construct the Eichler-Shimura map (which will evidently recover our construction from before) using this framework.

Definition 5.22 (Modular Forms)**.** Let $k \geq 2$ be an integer and $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group of the first kind. Then, the space of **modular forms of weight k with respect to Γ** (with coefficients in $\mathbb{C}$) is the $\mathbb{C}$-vector space

$$\mathcal{M}_k(\Gamma, \mathbb{C}) := H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k}).$$

The space of **cusp forms** (of weight k with respect to Γ) is the subspace

$$\mathcal{S}_k(\Gamma, \mathbb{C}) := H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k}(-\text{cusps})).$$

Note by $\underline{\mathrm{KS}}_{X_\Gamma}$, so long as all cusps are regular, we can write these spaces as (for $k \geq 0$)

$$\mathcal{M}_{k+2}(\Gamma, \mathbb{C}) = H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes(k+2)}) = H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k} \otimes \Omega^1_{X_\Gamma}(\text{cusps}))$$

and

$$\begin{aligned} \mathcal{S}_{k+2}(\Gamma, \mathbb{C}) &= H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes(k+2)}(-\text{cusps})) \\ &= H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k}(-\text{cusps}) \otimes \Omega^1_{X_\Gamma}(\text{cusps})) \\ &= H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k} \otimes \Omega^1_{X_\Gamma}). \end{aligned}$$

This is nice, because we can utilize the short exact sequence (1) for $X = Y_\Gamma$ to produce the Eichler-Shimura map. Note after descent, we have a short exact sequence (3)

$$0 \to \omega_{Y_\Gamma} \xrightarrow{\iota} \mathcal{U}_{Y_\Gamma} \otimes_{\underline{\mathbb{R}}} \mathcal{O}_{Y_\Gamma} \to \omega_{Y_\Gamma}^{-1} \to 0.$$

For notational simplicity, we henceforth replace all subscripts Y_Γ with just Γ, i.e., we denote $\omega_\Gamma := \omega_{Y_\Gamma}$, $\mathcal{U}_\Gamma := \mathcal{U}_{Y_\Gamma}$, $\Omega^1_\Gamma := \Omega^1_{Y_\Gamma}$, and $\mathcal{O}_\Gamma := \mathcal{O}_{Y_\Gamma}$.

But note from the construction of Equation 1 that this sequence is exact from the Hodge decomposition for $n = 1$, so in particular, it is locally split. Therefore, taking the k^{th} tensor power preserves exactness, and we have an injection

$$\iota^k : \omega_\Gamma^{\otimes k} \hookrightarrow \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \mathcal{O}_\Gamma.$$

Furthermore, Ω^1_Γ is a line bundle on Y_Γ, hence locally free, so tensoring by it preserves exactness. This means we get an inclusion

$$\iota^k \otimes \mathrm{id}_{\Omega^1} : \omega_\Gamma^{\otimes k} \otimes \Omega^1_\Gamma \hookrightarrow \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_\Gamma.$$

This induces an inclusion on the global sections. The long exact sequence for de Rham cohomology on X_Γ further gives a connecting map $H^0(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_\Gamma) \xrightarrow{\delta} H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$, hence in total we have the composition

$$H^0(Y_\Gamma, \omega_\Gamma^{\otimes k} \otimes \Omega^1_\Gamma) \xhookrightarrow{\iota^k \otimes \mathrm{id}} H^0(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_\Gamma) \xrightarrow{\delta} H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}).$$

The $\otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}$ at the end means the H^1 on the right is equipped with an action of complex conjugation. Applying complex conjugation to the first term (the space of meromorphic modular forms) gives us a map $\overline{H^0(Y_\Gamma, \omega_\Gamma^{\otimes k} \otimes \Omega^1_\Gamma)} \to H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$, hence we get the map

$$H^0(Y_\Gamma, \omega_\Gamma^{\otimes k} \otimes \Omega^1_\Gamma) \oplus \overline{H^0(Y_\Gamma, \omega_\Gamma^{\otimes k} \otimes \Omega^1_\Gamma)} \to H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}).$$

Pre-composing with the natural inclusion $H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k} \otimes \Omega^1_{X_\Gamma}) \hookrightarrow H^0(Y_\Gamma, \omega_\Gamma^{\otimes k} \otimes \Omega^1_{Y_\Gamma})$ gives us the Eichler-Shimura map

$$\mathrm{ES}_\Gamma : \mathcal{S}_{k+2}(\Gamma, \mathbb{C}) \oplus \overline{\mathcal{S}_{k+2}(\Gamma, \mathbb{C})} \to H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}). \tag{4}$$

As in 1.2, one can define the interior cohomology group $H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}) \subset H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$. We justify that, as in our original statement of the Eichler-Shimura isomorphism, the image of ES_Γ lives in the interior cohomology part.

Lemma 5.23. *Given the map* ES_Γ *above, its image is contained in* $H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$.

Proof. By identifying our local system $\mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma$ with $\underline{L(n, \mathbb{R})}_\Gamma$ (amounting to choosing a basis of $\mathbb{R}^2$ locally, which provides a basis everywhere via action by $\mathrm{SL}_2(\mathbb{R})$), we can recover the explicit form of f as an element in $H^0(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_\Gamma)$ as in §2.

$$f \mapsto \omega_f := (X - zY)^k \otimes f(z)\, dz.$$

The connecting map for de Rham cohomology is explicitly given by

$$\begin{aligned} H^0(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \Omega^1_\Gamma) &\xrightarrow{\delta} H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}) \\ \omega_f &\mapsto \left(c \mapsto \int_c \omega_f \right). \end{aligned}$$

For each cusp $s \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})$, denote D_s^* as some sufficiently small punctured open disk around s in Y_Γ. The connecting map commutes with restriction, so to show $\delta(\omega_f) \in H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$, it suffices to show that $\delta\left(\omega_f|_{D_s^*}\right)$ vanishes for each cusp s. But $H_1(D_s^*, \mathbb{Z}) = \pi_1(D_s^*)$ is generated by a loop around s, hence for some loop σ generating $\pi_1(D_s^*)$, it suffices to show that $\int_\sigma \omega_f = 0$.

Let $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma s = i\infty$. Since s is assumed to be a regular cusp, we have $\gamma \Gamma_s \gamma^{-1} = \left(\begin{smallmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$ for some $h > 0$. Thus, after sending s to $i\infty$ via conjugation by γ, we have that a loop around $s = i\infty$ generating $\pi_1(D_s^*)$ must lift in $\mathbb{H}$ to a path from $a + ib$ to $a + h + ib$ for some large enough $b \gg 0$ and $a \in \mathbb{R}$. But since f is a cusp form, it decays exponentially as $b \to \infty$, hence

$$\lim_{b \to \infty} \int_{a+ib}^{a+h+ib} \omega_f = 0$$

and the conclusion follows. ☐

With this, we can restrict the image of ES_Γ to the interior cohomology. By doing so, we claim that the map is an isomorphism.

Theorem 5.24 (Eichler-Shimura isomorphism)**.** *The map*

$$\mathrm{ES}_\Gamma : \mathcal{S}_{k+2}(\Gamma, \mathbb{C}) \oplus \overline{\mathcal{S}_{k+2}(\Gamma, \mathbb{C})} \to H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}})$$

is an isomorphism.

After translating this map to the "classical" setting in the first half of this exposition, one can prove injectivity as in §3. We concern ourselves, then, with proving surjectivity in the final section.

6. Proof of Surjectivity, Version 2

Given injectivity, it suffices to prove the dimensions (over $\mathbb{R}$) on both sides agree, or in other words

$$\begin{aligned}
4\dim_{\mathbb{C}} H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k} \otimes \Omega^1_{X_\Gamma}) &= 4\dim_{\mathbb{C}} \mathcal{S}_{k+2}(\Gamma, \mathbb{C}) \\
&= \dim_{\mathbb{R}} \left(\mathcal{S}_{k+2}(\Gamma, \mathbb{C}) \oplus \overline{\mathcal{S}_{k+2}(\Gamma, \mathbb{C})}\right) \\
&\overset{?}{=} \dim_{\mathbb{R}} H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}) \\
&= 2\dim_{\mathbb{C}} H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}) \\
&= 2\dim_{\mathbb{R}} H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma).
\end{aligned}$$

The advantage of having defined modular forms as some zeroth cohomology is that computing its dimension is immediate from Riemann–Roch.

Theorem 6.1 (Dimension of Space of Cusp Forms)**.** *Let g and c denote the genus and number of cusps of X_Γ, respectively. For $k \geq 0$, we have*

$$\dim_{\mathbb{C}} \mathcal{S}_{k+2}(\Gamma, \mathbb{C}) = (k+1)(g-1) + \frac{kc}{2}.$$

Proof. It is standard that $\deg \Omega^1_{X_\Gamma} = 2g-2$ (it is the canonical bundle, since X_Γ is a curve). From the Kodaira-Spencer map $\underline{\mathrm{KS}}_{X_\Gamma} : \omega_{X_\Gamma}^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{X_\Gamma}(\text{cusps})$, we see that

$$\begin{aligned}
2\deg \omega_{X_\Gamma} &= 2g-2+c \\
\implies \deg \omega_{X_\Gamma}^{\otimes k} &= k\left(g-1+\frac{c}{2}\right) \\
\implies \deg\left(\omega_{X_\Gamma}^{\otimes k} \otimes \Omega^1_{X_\Gamma}\right) &= (k+2)(g-1) + \frac{kc}{2}.
\end{aligned}$$

When $\deg \omega_{X_\Gamma} > 0$, Riemann–Roch gives us the desired result, so it suffices to show that this inequality must be true.

Since $c \geq 0$, we see that at best, $\deg \omega_{X_\Gamma} \geq -1$. If $\deg \omega_{X_\Gamma} = -1$, then $g = 0$ and $c = 0$, which is only possible if $X_\Gamma \simeq \mathbb{P}^1_{\mathbb{C}}$. But we know X_Γ admits $\mathbb{H}$ as its universal cover by construction, so this is not possible. If $\deg \omega_{X_\Gamma} = 0$, then we either have $(g, c) \in \{(1,0), (0,2)\}$. The latter situation implies $X_\Gamma \simeq \mathbb{P}^1_{\mathbb{C}}$ again and Y_Γ is the sphere with two points removed. We have $\pi_1(Y_\Gamma) \simeq \mathbb{Z}$, but this should agree with Γ modulo its elliptic and parabolic elements, all of which have finite order. But $\mathbb{Z}$, embedded into $\mathrm{SL}_2(\mathbb{R})$ as $\left(\begin{smallmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right)$, does not have finite covolume in $\mathrm{SL}_2(\mathbb{R})$, so this cannot be possible. Finally, if $(g, c) = (1, 0)$, then X_Γ is an elliptic curve, whose universal cover is $\mathbb{C}$ (namely, not $\mathbb{H}$). □

Corollary 6.2. *There are no modular forms of negative weight, i.e., for $k < 0$,*

$$H^0(X_\Gamma, \omega_{X_\Gamma}^{\otimes k}) = 0.$$

Proof. This is a direct consequence of $\deg \omega_{X_\Gamma} > 0$, as established in the proof above. $\square$

We now turn our attention to the interior cohomology group. It remains to show

$$\dim_{\mathbb{R}} H^1_!(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) = 2(k+1)(g-1) + kc.$$

Remark 6.3 (Notation). From now on, we denote h^i_* (for $* = c, !, \emptyset$) as $\dim_{\mathbb{R}} H^i_*$. Furthermore, as all sheaves will be the local system $\mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma$, we will suppress it, so $h^i_*(X)$ should be understood as $\dim_{\mathbb{R}} H^i_*(X, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma)$ unless otherwise specified.

Recall from §1.2 that we have the long exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0_c(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) & \longrightarrow & H^0(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) & \longrightarrow & \bigoplus_{s \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^0(D^*_s, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \\ \longrightarrow & H^1_c(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) & \longrightarrow & H^1(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) & \longrightarrow & \bigoplus_{s \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^1(D^*_s, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \\ \longrightarrow & H^2_c(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) & \longrightarrow & H^2(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) & \longrightarrow & \bigoplus_{s \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} H^2(D^*_s, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \end{array}$$

We will boil our proof down to three lemmata. We state them below, then show how they imply our desired equality on $h^1_!(Y_\Gamma)$, then prove each lemma separately.

Lemma 6.4. *Let $\chi(X, \mathcal{F})$ be the Euler characteristic of $\mathcal{F}$ as a sheaf on X. Then,*

(1) $\chi(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) = (k+1)(2-2g-c)$,
(2) $h^0(D^*_s) = h^1(D^*_s) = 1$, and
(3) $h^0(Y_\Gamma) = h^2(Y_\Gamma) = h^2_c(Y_\Gamma) = 0$.

Assuming Lemma 6.4, the proof of surjectivity goes as follows. We seek to compute $h^1_!(Y_\Gamma) = h^1_c(Y_\Gamma) - \sum_s h^0(D^*_s)$. From (3), we can truncate the cohomology long exact sequence and get

$$\sum_s h^0(D^*_s) - h^1_c(Y_\Gamma) + h^1(Y_\Gamma) - \sum_s h^1(D^*_s) = 0.$$

From (2), this implies $h^1_c(Y_\Gamma) = h^1(Y_\Gamma)$. Thus, we have

$$\begin{aligned} &\text{(from above)} && h^1_c(Y_\Gamma) = h^1(Y_\Gamma) \\ &\text{(by (1))} && \qquad\quad = h^0(Y_\Gamma) + h^2(Y_\Gamma) + (k+1)(2g-2+c) \\ &\text{(by (3))} && \qquad\quad = (k+1)(2g-2+c), \end{aligned}$$

and then (2) again gives us

$$h^1_!(Y_\Gamma) = h^1_c(Y_\Gamma) - \sum_s h^0(D^*_s) = (k+1)(2g-2+c) - c = 2(k+1)(g-1) + kc$$

as desired.

Proof of (1). We first show $\chi(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) = (k+1)\chi(Y_\Gamma, \underline{\mathbb{R}})$, then show $\chi(Y_\Gamma, \underline{\mathbb{R}}) = 2 - 2g - c$.

We construct a nice triangulation $\mathfrak{T}$ of X_Γ as in the beginning of the proof of Proposition 4.4, in particular one where all cusps are included among the vertices. We then consider the open cover $U_\bullet = \{U_i\}_{i \in I}$ of Y_Γ consisting of the following open subsets:

- the interiors of all triangles in $\mathfrak{T}$
- pairwise disjoint tubular neighborhoods of each edge of $\mathfrak{T}$ with the vertices removed
- small open balls around each non-cusp vertex in $\mathfrak{T}$

As all opens and their finite intersections are contractible, $U_\bullet$ is a "good" cover for Čech cohomology. Note that, as expected from H^3 vanishing on a curve, any intersection between four open sets is empty, so the associated Čech complex only has terms up to degree 2. By a standard comparison theorem between sheaf and Čech cohomology, we have

$$\begin{aligned}\chi(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) &= h^0(Y_\Gamma) - h^1(Y_\Gamma) + h^2(Y_\Gamma) = \check{h}^0(U_\bullet) - \check{h}^1(U_\bullet) + \check{h}^2(U_\bullet) \\ &= \dim_{\mathbb{R}} C^0(U_\bullet, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) - \dim_{\mathbb{R}} C^1(U_\bullet, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \\ &\qquad + \dim_{\mathbb{R}} C^2(U_\bullet, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma)\end{aligned}$$

where $\check{h}^i(U_\bullet) := \dim_{\mathbb{R}} \check{H}^i(U_\bullet, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma)$ and $C^\bullet(U_\bullet, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma)$ is the Čech complex for $\mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma$ with respect to the open cover $U_\bullet$. But note the Čech complex only sees the sheaf locally, so replacing $\mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma$ with $\underline{\mathbb{R}}^{k+1}$ would not change the terms of the Čech complex (although it would change the chain maps). In particular, we have

$$\begin{aligned}\chi(Y_\Gamma, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) &= \sum_{i=0}^{2} (-1)^i \check{h}^i(U_\bullet) \\ &= \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{R}} C^i(U_\bullet, \mathrm{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \\ &= \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{R}} C^i(U_\bullet, \underline{\mathbb{R}}^{k+1}) \\ &= \sum_{i=0}^{2} (-1)^i (k+1) \dim_{\mathbb{R}} C^i(U_\bullet, \underline{\mathbb{R}}) \\ &= (k+1)\chi(Y_\Gamma, \underline{\mathbb{R}}).\end{aligned}$$

It is standard that since X_Γ is a compact Riemann surface, its Euler characteristic is $\chi(X_\Gamma, \underline{\mathbb{R}}) = 2 - 2g$. To prove $\chi(Y_\Gamma, \underline{\mathbb{R}}) = 2 - 2g - c$, it thus suffices to show that removing a cusp decreases the Euler characteristic by 1.

The Mayer-Vietoris sequence on the open cover $X_\Gamma = (X_\Gamma \smallsetminus \{s\}) \cup D_s$, for some small open disk D_s around a cusp s, gives us the relation

$$\chi(X_\Gamma \smallsetminus \{s\}) + \chi(D_s) = \chi(X_\Gamma) + \chi(D_s^*).$$

Note that D_s^* is homotopy equivalent to the unit circle S^1, which has Euler characteristic 0. It is also standard that $\chi(D_s) = 1$, so it follows that

$$\chi(X_\Gamma) = \chi(X_\Gamma \smallsetminus \{s\}) + 1,$$

as desired. □

Proof of (2). We run the same argument in the above proof for D_s^*, namely that the Euler characteristic of a local system only depends on the rank of the local system, to obtain

$$\chi(D_s^*, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) = h^0(D_s^*) - h^1(D_s^*) + h^2(D_s^*) = (k+1)\chi(D_s^*, \underline{\mathbb{R}}) = 0,$$

where the last equality follows from D_s^* being homotopy equivalent to a unit circle, which has Euler characteristic 0. Any H^2 of the unit circle will vanish, hence we are left with $h^0(D_s^*) = h^1(D_s^*)$.

Finally, we can deduce $h^0(D_s^*) = 1$ from Lemma 4.7, which in the case $G = \pi_1(D_s^*) = \mathbb{Z}$, $X = \mathbb{R}$, and $\pi : \mathbb{R} \twoheadrightarrow D_s^* \simeq S^1$ the universal cover of D_s^* gives the isomorphisms $H^i(\pi_1(D_s^*), \operatorname{Sym}^k \mathbb{R}^2) \simeq H^i(D_s^*, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma)$. We clarify that the $\pi_1(D_s^*)$-action on $\operatorname{Sym}^k \mathbb{R}^2$ is induced by the $\mathbb{Z}$-action on $\mathbb{R}^2$, where we identify $\mathbb{Z}$ with the stabilizer $\left(\begin{smallmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \subset \mathrm{SL}_2(\mathbb{R})$ of s. But this action clearly has just a one-dimensional invariant subspace for the induced action on $\operatorname{Sym}^k \mathbb{R}^2$, hence for $i = 0$, we deduce $h^0(D_s^*, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) = 1$. □

Proof of (3). For a local system $\mathcal{L}_\Gamma$ of finite-dimensional $\mathbb{R}$-vector spaces on Y_Γ, the cup product

$$H^i(Y_\Gamma, \mathcal{L}_\Gamma) \otimes H_c^{2-i}(Y_\Gamma, \mathcal{L}_\Gamma^\vee) \to H_c^2(Y_\Gamma, \underline{\mathbb{R}}) \overset{\mathrm{Tr}}{\simeq} \mathbb{R}$$

is perfect, as it is perfect locally due to Poincaré duality. For $\mathcal{L}_\Gamma = \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma$, we can use the pairing B in the proof of injectivity (§3) to deduce that $\operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma$ is self-dual, hence we have that

$$H^2(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \simeq H_c^0(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \subset H^0(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) \simeq H_c^2(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma).$$

Thus, it remains to prove $H^0(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma) = 0$ for $k > 0$. From the proof of (2), this is the space of Γ-invariants of $\operatorname{Sym}^k \mathbb{R}^2 \simeq L(n, \mathbb{R})$. For the rest of the proof, we will utilize the perfect pairing $B : L(n, \mathbb{C}) \times L(n, \mathbb{C}) \to \mathbb{C}$ given in the proof of injectivity, §3. From now on, we identify $\operatorname{Sym}^k \mathbb{R}^2$ with $L(n, \mathbb{R})$. Choose some $v \in L(n, \mathbb{R})^\Gamma = H^0(Y_\Gamma, \operatorname{Sym}^k_{\underline{\mathbb{R}}} \mathcal{U}_\Gamma)$. We define the complex-valued polynomial

$$p(z) \coloneqq B(v \otimes (zX + Y)^k), \qquad z \in \mathbb{C}.$$

Since B is a perfect pairing, in order to force $v = 0$, it suffices to show $p \equiv 0$. Following Corollary 6.2, it suffices to show $p(z)$ is a modular form with respect to Γ of negative weight. As $p(z)$ is just a polynomial, it is clearly holomorphic. It thus remains to check (a) $p(z)$ satisfies the automorphy condition and (b) it is bounded at the cusps.

Choose some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. By construction, we have $v = \gamma v$. Furthermore, one can show explicitly (as we did in class) that B is $\mathrm{SL}_2(\mathbb{R})$-equivariant, hence we have

$$\begin{aligned} p(\gamma z) &= B\left(\gamma v \otimes ((\gamma z)X + Y)^k\right) \\ &= B\left(v \otimes \left((\gamma z)\gamma^{-1}X + \gamma^{-1}Y\right)^k\right) \\ &= (cz+d)^{-k} B(v \otimes ((az+b)(dX - cY) + (cz+d)(-bX + aY))^k) \\ &= (cz+d)^{-k} B(v \otimes (zX + Y)^k) \\ &= (cz+d)^{-k} p(z), \end{aligned}$$

demonstrating (a). The boundedness condition (b) follows from the facts that $p(z)$ is a polynomial yet, by (a), it is invariant under a translation $z \mapsto z + h$ (where $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ generates the stabilizer of $i\infty$ in Γ). In fact, this forces $p(z)$ to be constant, and the automorphy condition requires this constant to be 0. □

References

[AS86] Avner Ash and Glenn Stevens. "Modular forms in characteristic ℓ and special values of their L-functions". In: *Duke Mathematical Journal* 53.3 (1986), pp. 849–868. DOI: 10.1215/S0012-7094-86-05347-7.

[Bel21] Joël Bellaïche. *The Eigenbook: Eigenvarieties, Families of Galois Representations, p-adic L-functions*. Pathways in Mathematics. Birkhäuser, 2021. ISBN: 978-3-030-77262-8. DOI: 10.1007/978-3-030-77263-5.

[Con05] Brian Conrad. "Modular Forms, Cohomology, and the Ramanujan Conjecture". Unpublished book draft, available at http://virtualmath1.stanford.edu/~conrad/modbook.pdf. 2005.

[Del71] Pierre Deligne. "Formes modulaires et représentations ℓ-adiques". In: *Séminaire Bourbaki, 21e année (1968/69), Exposé 355*. Vol. 179. Lecture Notes in Mathematics. Springer, 1971, pp. 139–172.

[DM68] P. H. Doyle and D. A. Moran. "A Short Proof that Compact 2-Manifolds Can Be Triangulated". In: *Inventiones Mathematicae* 5 (1968), pp. 160–162. DOI: 10.1007/BF01405244. URL: https://webhomes.maths.ed.ac.uk/~v1ranick/papers/doylemoran.pdf.

[Gro57] Alexander Grothendieck. "Sur quelques points d'algèbre homologique". In: *Tôhoku Mathematical Journal* 9.2 (1957), pp. 119–221. DOI: 10.2748/tmj/1178244839.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. New York: Springer-Verlag, 1977. ISBN: 978-0-387-90244-9.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002. URL: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[Hid93] Haruzo Hida. *Elementary Theory of L-functions and Eisenstein Series*. Vol. 26. London Mathematical Society Student Texts. Cambridge: Cambridge University Press, 1993.

[KM85] Nicholas M. Katz and Barry Mazur. *Arithmetic Moduli of Elliptic Curves*. Vol. 108. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 1985. ISBN: 978-0-691-08349-1.

[Lev11] Brandon Levin. *Modular Symbols*. Notes from Brian Conrad's number theory learning seminar, Stanford University. 2011. URL: https://virtualmath1.stanford.edu/~conrad/DarmonCM/2011Notes/Modular%20Symbols.pdf.

[Mum74] David Mumford. *Abelian Varieties*. Vol. 5. Tata Institute of Fundamental Research Studies in Mathematics. With appendices by C. P. Ramanujam and Yuri Manin. Bombay: Oxford University Press, 1974.

[Shi71] Goro Shimura. *Introduction to the Arithmetic Theory of Automorphic Functions*. Vol. 11. Publications of the Mathematical Society of Japan. Reprinted by Iwanami Shoten and Princeton University Press. Princeton, NJ: Princeton University Press, 1971.

[Sta25] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2025.

[Ste43] N. E. Steenrod. "Homology with Local Coefficients". In: *Annals of Mathematics*. Second Series 44.4 (1943), pp. 610–627. DOI: 10.2307/1968963.

[Vak25] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. Princeton University Press, 2025, p. 704. ISBN: 9780691268668.